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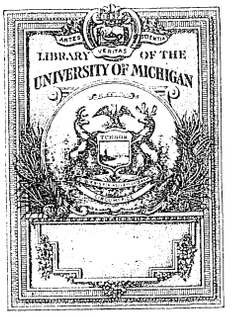

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FOR SCHOOLS AND COLLEGES

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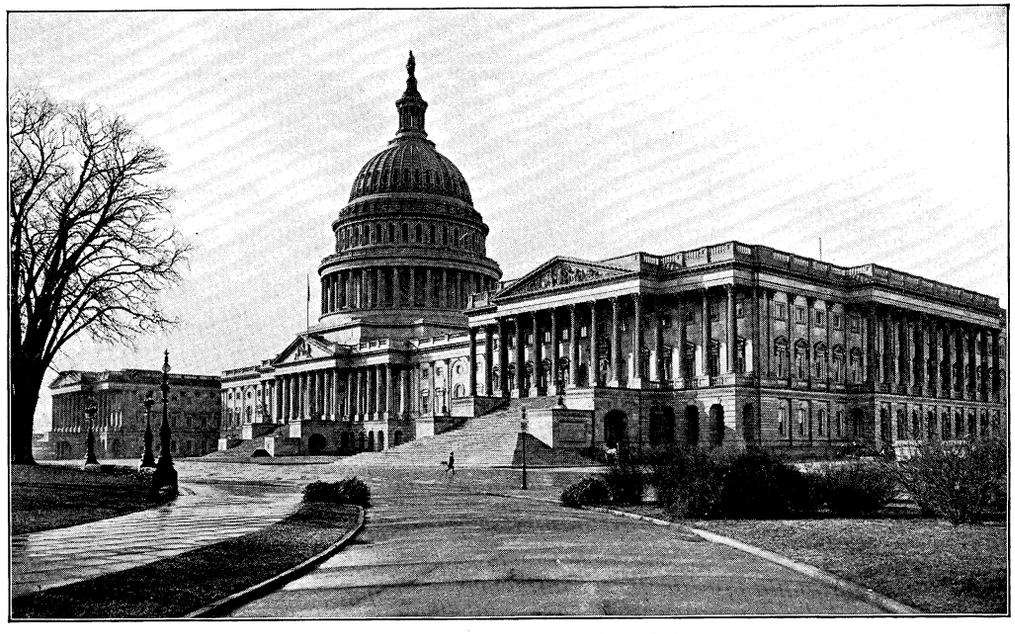

NATIONAL CAPITOL, WASHINGTON, D.C.
This building illustrates the use of geometrical forms in architecture

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SOLID GEOMETRY

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PREFACE

The general principles underlying the preparation of this text have been previously stated in the preface of the authors' "Plane Geometry." The attempt has been made to harmonize modern demands with the best traditions of the subject. This policy involves, among other things, greater attention to motive in the presentation of each topic, and, as well, certain deviations from the usual order of theorems, with care in the choice and distribution of the exercises.

The following features are called to the attention of the reader:

1. *A brief preliminary course precedes the demonstrative work.* This plan, based on ample experience, aims to introduce work on actual solids at the earliest moment, and so to prepare more adequately for "three-dimensional thinking." If desired, these pages (pp. 330–338) may be studied in connection with the propositions of Book VI.

2. *The arrangement of topics brings the essentials of the subject into the foreground.* The list of theorems is reduced to a safe minimum. The suggestions and requirements of the National Geometry Syllabus prepared by the Committee of Fifteen have been kept in mind throughout, together with the requirements of various examining boards. For reasons of economy, primarily, prisms and cylinders are treated together. This is also true of pyramids and cones. For similar reasons, polyhedral angles are studied in connection with spherical polygons. A number of familiar theorems are thus rendered unnecessary or are given as corollaries. The early introduction, in Book VIII, of the mensuration of the sphere serves to

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separate the treatment of the sphere as a solid from spherical geometry proper.

3. *Geometric drawing, in connection with the principles of projection, receives more than ordinary attention.* In the opinion of the authors, even a modest training in this direction is valuable both in the demonstration of theorems and in other branches of pure and applied mathematics.

4. *The exercises are not excessive in number or difficulty, but emphasize all types — constructions, computations, and original theorems.* The vexatious question of applied problems is approached in a conservative spirit. Technicalities have, as far as possible, been avoided in these problems. Many of the problems are suggestive and may easily be adapted to local conditions. Solid Geometry offers an unusual field for the introduction of "community problems." The use of squared paper is encouraged. As in the Plane Geometry (pp. 259–264) a few problems requiring trigonometric solution are introduced. For their solution a table of natural functions is added. The treatment of loci, while largely informal, may be made the basis for the careful development of space intuition. Special attention is called to the use of *composite figures* (see pp. 330–338, 367, 383, 423, 429, 436–438, 455).

5. *Considerations of limits are usually preceded by an informal discussion or development.* A serious effort has been made to reduce the formal treatment to its simplest terms.

6. *Illustrations, discussions, and historical notes are interspersed throughout the text, in the hope that they may add to the student's interest.*

The authors recommend, especially in the case of young students, an extensive use of simple models, at least in the beginning. A few sticks or pieces of wire, together with some rectangular pieces of cardboard or plate glass, will serve to make concrete any proposition in Book VI. The demonstrations should first be given in connection with such improvised models, before blackboard diagrams are substituted. Wire

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models of solids are easily made. Interior lines or sections can be indicated in these models with strings or elastic cords. The value of blackboard figures is often increased by the use of colored crayons. Pupils should be led by slow degrees to depend on mental diagrams. They should be requested to summarize proofs, orally and also in writing. The preparation of outlines both of topics and of chapters is a very valuable exercise.

The authors acknowledge their indebtedness to the following gentlemen, who read portions of the manuscript and offered valuable criticisms or suggestions: Mr. F. E. Crofts, of San Francisco, Cal.; Mr. E. H. Barker, of Los Angeles, Cal.; Dr. H. W. Stager, of Fresno, Cal.; and Mr. A. E. Booth, of New Haven, Conn. They also wish to thank their colleagues in Rochester and Newark for their interest in the preparation of this text and for their assistance in using the manuscript in their classes. Moreover, the standard European and American texts were frequently consulted. The figure accompanying the note on page 337 was suggested by the text of Messrs. Godfrey and Siddons, while the cut on page 395 is due to Professor F. Schilling.

THE AUTHORS

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SYMBOLS AND ABBREVIATIONS

+	plus, or added to.	adj.	adjacent.
−	minus, or diminished by.	alt.	alternate.
=	equal, or is equal to.	ax.	axiom.
>	is greater than.	circum.	circumference.
<	is less than.	comp.	complement, or complementary.
≡	is congruent to.	cons.	construction.
∴	therefore, or hence.	cor.	corollary.
⊥	perpendicular, or is perpendicular to.	corr.	corresponding.
		def.	definition.
⊥s	perpendiculars.	ex.	exercise.
∥	parallel, or is parallel to.	ext.	exterior.
∥s	parallels.	hom.	homologous.
∽	is similar to, or similar.	hyp.	hypothesis.
∠	angle.	int.	interior.
∠s	angles.	isos.	isosceles.
△	triangle.	prop.	proposition.
△s	triangles.	rect.	rectangle.
▱	parallelogram.	rt.	right.
▱s	parallelograms.	sq.	square.
⊙	circle.	st.	straight.
⊙s	circles.	supp.	supplement, or supplementary.

a. s. a., having a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other.

s. a. s., having two sides and the included angle of one equal respectively to two sides and the included angle of the other.

s. s. s., having three sides of one equal respectively to three sides of the other.

rt. △, h. l., being right triangles and having the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other.

rt. △, h. a., being right triangles and having the hypotenuse and an adjoining angle of one equal respectively to the hypotenuse and an adjoining angle of the other.

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REFERENCES TO PLANE GEOMETRY

The following list is not intended to be complete, but includes the facts of plane geometry referred to most frequently in the following pages.

113. Two figures which may be made to coincide in all their parts are said to be *congruent.* We shall use the symbol $\equiv$ for the word "congruent." $F_1 \equiv F_2$ means that F_1 is congruent to F_2. From this definition it follows that *figures congruent to the same figure are congruent to each other.*

131. Preliminary Assumptions and Propositions on Lines and Angles:

Through a given point an indefinite number of straight lines may be drawn.

Two straight lines can intersect in but one point.

One and only one straight line can be drawn through two given points.

Two straight lines do not inclose a space.

All right angles are equal.

At a given point in a given line only one perpendicular can be drawn to that line (in the same plane).

Vertical angles are equal.

At a given point in a given line a line may be drawn making with the given line an angle equal to a given angle.

132. General Axioms:

1. Magnitudes which are equal to the same magnitude, or to equal magnitudes, are equal to each other.

In other words, a magnitude may be substituted for its equal.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

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4. If equals are multiplied by equals, the products are equal.

5. If equals are divided by equals, the quotients are equal.

The divisor must not be zero.

6. Like powers or like positive roots of equals are equal.

7. The whole of any magnitude is equal to the sum of all its parts.

8. The whole of any magnitude is greater than any part of it

AXIOM OF SUPERPOSITION. Any figure may be moved about in space without changing either its size or its shape.

133. If two triangles have a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other, the triangles are congruent. (a. s. a.)

134. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent. (s. a. s.)

135. Homologous parts (sides or angles) of congruent triangles are those parts which are opposite parts known to be equal.

138. If two sides of a triangle are equal, the angles opposite those sides are equal.

139. An equilateral triangle is also equiangular.

140. If two angles of a triangle are equal, the sides opposite those angles are equal.

141. An equiangular triangle is also equilateral.

142. If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent. (s. s. s.)

144. CONGRUENCE OF POLYGONS. A polygon of four sides has eight parts, one of five sides ten parts, and, in general, one of n sides has $2n$ parts, n sides and n angles. It can be proved by superposition that if two polygons of n sides have $2n - 3$

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consecutive parts of one respectively equal to $2n - 3$ consecutive parts of the other, the polygons are congruent.

150. If two isosceles triangles are constructed on the same base, the line that joins their vertices bisects the common base at right angles.

151. Two points each equidistant from the extremities of a line determine the perpendicular bisector of that line.

161. Of all lines that can be drawn from a given point to a given line, only one is perpendicular to the given line.

165. If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, the triangles are congruent. (rt.$\triangle$ h. l.)

166. If two right triangles have the hypotenuse and an adjoining angle of one equal respectively to the hypotenuse and an adjoining angle of the other, the right triangles are congruent. (rt.$\triangle$ h. a.)

170. Parallel lines are lines which lie in the same plane and do not meet, however far they are produced.

When two straight lines are cut by a transversal, if the alternate-interior angles are equal, the straight lines are parallel.

171. When two straight lines are cut by a transversal, if

(a) the alternate-exterior angles are equal; or if

(b) the exterior-interior angles are equal; or if

(c) any two interior or any two exterior angles on the same side of the transversal are supplementary,

the two straight lines are parallel.

172. Two straight lines in the same plane perpendicular to the same straight line are parallel.

174. Parallel Axiom. Through a given point but one straight line can be drawn parallel to a given straight line.

Two intersecting straight lines cannot both be parallel to the same straight line.

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175. Two straight lines in the same plane parallel to the same straight line are parallel to each other.

176. If two parallel lines are cut by a transversal:

(a) the alternate-interior angles are equal;
(b) the exterior-interior angles are equal;
(c) the alternate-exterior angles are equal;
(d) any two interior or any two exterior angles on the same side of the transversal are supplementary.

178. If two lines are perpendicular respectively to two intersecting lines, they cannot be parallel.

180. The sum of the interior angles of a triangle equals a straight angle.

194. The sum of the interior angles of a polygon is equal to as many straight angles as the figure has sides less two.

195. If the sides of a polygon are produced in succession, the sum of the exterior angles thus formed equals two straight angles.

196. A parallelogram is the quadrilateral inclosed when two pairs of parallel lines intersect each other.

201. The diagonal of a parallelogram divides it into two congruent triangles, and the opposite sides of a parallelogram are equal.

202. Parallel lines included between parallel lines are equal.

204. The diagonals of a parallelogram bisect each other.

205. A rectangle is a parallelogram whose angles are right angles.

207. A square is an equilateral rectangle.

208. If a quadrilateral has (*a*) its opposite sides equal, or (*b*) one pair of opposite sides equal and parallel, or (*c*) its diagonals bisecting each other, or (*d*) its opposite angles equal, it is a parallelogram.

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215. A trapezoid is a quadrilateral two and only two of whose sides are parallel. The nonparallel sides of a trapezoid are called the legs, and the parallel sides the bases.

216. The line joining the mid-points of the legs is called the mid-line of the trapezoid.

219. The mid-line of a trapezoid is parallel to the bases and equal to half their sum.

222. Axioms of Inequality. Proofs involving inequalities presuppose the following axioms (see § 132).

8. The whole of any magnitude is greater than any part of it.

9. If equals are added to unequals, the results are unequal in the same sense.

10. If equals are subtracted from unequals, the remainders are unequal in the same sense.

11. If unequals are subtracted from equals, the results are unequal in the opposite sense.

12. If of three magnitudes the first is greater than the second, and the second is greater than the third, then the first is greater than the third.

224. If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.

225. If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.

227. Of all lines that can be drawn from a given point to a given line, the perpendicular is the shortest.

228. In any triangle the sum of two sides is greater than the third side.

232. If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

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233. Of two straight lines drawn from the same point in a perpendicular to a given line and cutting off on the line unequal segments from the foot of the perpendicular, the more remote is the greater.

239. All points in the perpendicular bisector of a line are equidistant from the extremities of the line; and, conversely, all points equidistant from the extremities of a line lie in the perpendicular bisector.

240. All points in the bisector of an angle are equidistant from the sides of the angle; and, conversely, all points equidistant from the sides of an angle lie in the bisector of the angle.

241–242. The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the vertices of the triangle.

This point is called the circumcenter of the triangle.

245–246. The bisectors of the angles of a triangle are concurrent in a point equidistant from the three sides.

This point is called the incenter of the triangle.

247–248. The medians of a triangle are concurrent in a point which is two thirds of the distance from each vertex to the middle of the opposite side.

This point is called the centroid of the triangle.

249-250. The altitudes of a triangle are concurrent in a point called the orthocenter of the triangle.

251. Preliminary Propositions on Circles:

A circle can be drawn about any given point as a center, with a radius equal to any given line.

Radii of the same circle are equal.

Circles which have equal radii are equal.

In the same circle or in equal circles:

Equal central angles intercept equal arcs.

Equal arcs are intercepted by equal central angles.

Equal chords subtend equal arcs.

Equal arcs are subtended by equal chords.

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A central angle is measured by its intercepted arc.

In the same circle or in equal circles the greater central angle intercepts the greater arc, and the greater arc is intercepted by the greater central angle.

A diameter of a circle bisects the circle.

252. A diameter perpendicular to a chord bisects the chord and the arcs subtended by it.

264. In the same circle or in equal circles equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.

270. A straight line perpendicular to a radius at its extremity is a tangent to the circle.

275. The tangents drawn to a circle from an external point are equal, and make equal angles with the line joining that point to the center.

280. If two circles intersect each other, the line of centers is perpendicular to their common chord at its middle point.

286. An angle inscribed in a semicircle is a right angle.

294. The locus of a point satisfying a given condition is the figure containing all the points that fulfill the given condition (or answer the given description), and no other points.

295. Rule for solving Locus Problems:

1. Locate a number of points which satisfy the given condition, and thus obtain a notion of what the locus is.
2. Prove that every point satisfying the given condition lies in the assumed locus.
3. Prove that every point of the assumed locus satisfies the given condition.

321. Fundamental Principle of Areas. The area of a rectangle is equal to the product of its base and altitude.

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322. Two rectangles are to each other as the products of their bases and altitudes.

326. The area of a right triangle is equal to one half the product of its legs.

327. The area of a kite (rhombus, square) is equal to one half the product of its diagonals.

328. The area of a parallelogram is equal to the product of its base and altitude.

329. Two parallelograms are to each other as the products of their bases and altitudes.

333. The area of a triangle is equal to half the product of its base and altitude.

334. If T and T' are the areas of two triangles with bases b and b', and altitudes h and h', then

(1) $\frac{T}{T'} = \frac{bh}{b'h'}$,

(2) $\frac{T}{T'} = \frac{b}{b'}[h = h']$,

(3) $\frac{T}{T'} = \frac{h}{h'}[b = b']$,

(4) $T = T'[h = h', b = b']$.

335. The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.

336. The area of a trapezoid is equal to the product of its altitude and mid-line.

337. If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including the equal angles.

344. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the legs.

358. In every proportion the product of the extremes is equal to the product of the means.

359. The mean proportional between two numbers is equal to the square root of their product.

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360. If the product of a pair of numbers equals the product of a second pair, the four numbers will be in proportion when written in any order that makes one pair the extremes and the other pair the means.

361. In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

362. If four numbers are in proportion, they are in proportion by alternation; that is, the first term is to the third term as the second is to the fourth.

363. If four numbers are in proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth is to the third.

364. If four numbers are in proportion, they are in proportion by addition; that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.

365. If four numbers are in proportion, they are in proportion by subtraction; that is, the difference of the first two terms is to the second term as the difference of the last two terms is to the fourth term.

370. Similar polygons are polygons which have their angles respectively equal and their homologous sides proportional.

372. If a line is drawn through two sides of a triangle, parallel to the third side, it divides those sides proportionally.

379. If two triangles have the angles of one equal respectively to the angles of the other, the triangles are similar.

386. If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.

389. If two triangles have their sides respectively proportional, they are similar.

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391. The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.

393–394. If in a right triangle a perpendicular is let fall from the vertex of the right angle upon the hypotenuse:

The perpendicular is the mean proportional between the segments of the hypotenuse.

Either leg is the mean proportional between the whole hypotenuse and the adjacent segment.

412. If two polygons are similar, they are composed of the same number of triangles, similar each to each and similarly placed.

415. The perimeters of two similar polygons are to each other as any two homologous sides.

416. The areas of two similar polygons are to each other as the squares of any two homologous sides.

431. A polygon that is both equiangular and equilateral is called a regular polygon.

A polygon whose sides are chords of a circle is called an inscribed polygon.

A polygon whose sides are tangent to a circle is called a circumscribed polygon.

456. The area of a regular polygon is equal to half the product of its apothem and its perimeter.

463. The circumference of a circle equals $2\pi r$.

464. The length of an arc of n degrees is $\frac{n}{360} \times 2\pi r$.

465. The area of a circle equals πr^2.

468. The area of a sector whose central angle contains n degrees is $\frac{n}{360} \times \pi r^2$.

469. The area of a sector is equal to one half the product of its radius and its arc.

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476. Limit of a Variable. When the successive values of a variable approach a certain constant number so that the difference between the constant and the variable becomes and remains less than any assigned positive number, however small, then the constant is called the limit of the variable.

485. The apothem of a regular inscribed polygon approaches the radius when the number of sides is indefinitely increased, and the limit of the ratio of the apothem and radius is unity.

487. The circumference of a circle is the common limit approached by the perimeters of a regular inscribed polygon and the similar regular circumscribed polygon, as the number of sides is continually increased.

488. The area of a circle is the common limit approached by the areas of these polygons.

489. If the variable x approaches the limit a, and if c is a constant, the product cx will approach the limit ca.

490. Two circumferences have the same ratio as their radii.

495. The areas of two circles are to each other as the squares of their radii or of their diameters.

---

499. Origin of Solid Geometry. It is claimed by ancient writers that geometry originated through surveying. This refers, of course, more especially to plane geometry. There can be little doubt, however, that the study of three-dimensional figures likewise arose from practical activities. Among these may be mentioned the building of houses, temples, monuments, embankments and canals, the making of household implements, pottery, furniture, jewelry, and the construction of wells and granaries.

It is true that the beginnings of these arts antedate all record. But the practice of uncivilized tribes of our own day suggests that primitive man was led by the natural resources at his command to imitate those solids which he found most

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useful for his purposes, and to invent new forms when he found none that suited him. This is especially true in building. The mountaineer needs a shelter different from that of the prairie dweller. The Hottentot, as well as the Eskimo, builds a hemispherical hut, the Indian erects a conical tepee, while the forest dweller constructs a round or a rectangular stockade with flat or sloping roof. And with the desire for more permanent buildings came the shaping of stones and of bricks, which gradually assumed the present rectangular form.

Thus at an early date man became practically familiar with the shapes and with some of the properties of the principal geometric solids — spheres, cylinders, cones, pyramids, rectangular solids, and prisms. It is known that the Egyptians had methods of determining the contents of wells and granaries. These "rules of thumb" were replaced at a much later date by a scientific mode of procedure.

500. Value of Solid Geometry. The world in which we live constantly presents to our attention a multitude of material objects of infinitely varied forms. Careful examination, however, reveals the fact that certain forms occur very frequently, and that others appear as approximations to these. A knowledge of these fundamental forms, therefore, is sufficient for all ordinary purposes and is most desirable as a basis for further study. Thus we readily observe the underlying rectangular form of houses, rooms, boxes, desks; the cylindrical form of pipes, wires, coins, towers, chimneys; the spherical form of balls, globes, domes — to mention only a few familiar examples. Nature exhibits the same forms in endless combinations and in beautiful symmetry, from the cylindrical form of a cornstalk and the wonderful regularity of numerous crystals to the spherical shape of the heavenly bodies.

A well-developed space intuition and an acquaintance with the principles of solid geometry are, moreover, of prime importance in drawing, mechanics, engineering, architecture, crystallography, astronomy, and optics.

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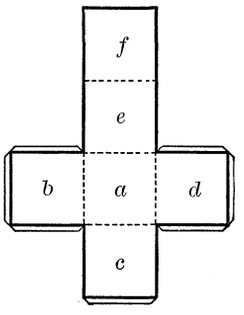
SOLID GEOMETRY

INTRODUCTION

501. Nature of Solid Geometry. All the figures of plane geometry are supposed to lie in a plane or flat surface. Solid geometry is concerned with the properties of figures all of whose points do *not* lie in the same plane. These figures are sometimes called **three-dimensional figures** from the fact that we associate with the most familiar solids ideas of length, breadth, and thickness or height.

502. There are two difficulties which confront the beginner in solid geometry. The first has to do with forming a mental image of a three-dimensional figure; the second with its representation on a flat surface, like the blackboard or a page of the notebook, and with interpreting correctly such a figure when it is already drawn or merely described in the text. To assist in overcoming these difficulties, the following exercises, informal in character, will give some practice in constructing and drawing solids and in understanding three-dimensional relations. The formal demonstrations of the underlying principles will be given later (Book VI).



503. Models. For a proper grasp of the principles of solid geometry the construction of models of a few simple solids is desirable. Cardboard models are easily made and will serve for ordinary purposes. A wire model of a solid is useful when it is necessary to point out lines lying in the interior of the solid.

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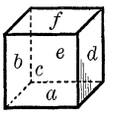
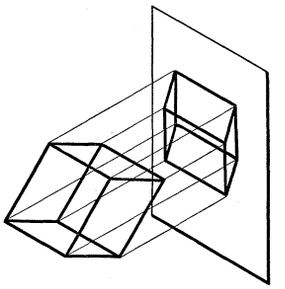
504. Model of a Cube. On a piece of cardboard or stiff paper draw the figure on the preceding page, consisting of six congruent squares. Make the sides of the squares about three inches long. By using the square a as a base, and by folding the squares b, c, d, e, about the sides of the square a, the square f can be brought directly over a, thus forming a bounded solid called a cube. (That such a solid exists, and that the squares will actually fit together, is assumed for the present.)

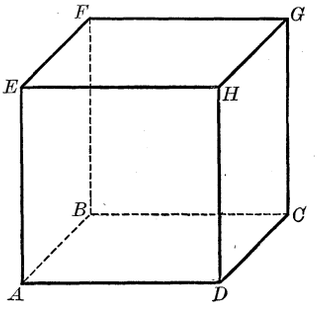


505. When a three-dimensional figure is represented by a drawing on a plane surface, not all of its parts (edges and bounding surfaces) appear in their original form and size.

If a wire model of a cube is placed between a bright light and a screen, a shadow of the model will appear on the screen. It will then be noted that some of the bounding surfaces of the cube appear as parallelograms or trapezoids instead of squares, and that not all of the edges appear as of equal length.



506. The picture on a plane surface of what is in reality a three-dimensioned figure is called a **projection** of that figure. The form of the projection of a solid depends not only on the shape of the solid, but also on the position of the solid with reference to the plane on which it is projected, and on the direction of the projecting rays.



507. A satisfactory projection (called a *cabinet projection*) of a cube can be made as follows: Draw a square $BCGF$ equal to one face of the cube. Through the vertices of this square, and making angles BCD and FGH each 45°, draw the four parallel lines

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CD, *GH*, *BA*, and *FE*, each half the length of one side of the square. Join in succession the extremities of these lines, forming a second square. It will be noted that the cube is supposed to be placed so that one face is parallel to the plane of the paper.

Points on the surface or in the interior of the cube can then be determined in the drawing by reference to the vertices.

508. The demonstrations of solid geometry must be based to a large extent on those of plane geometry. In plane geometry, however, all the elements of the figures considered are situated in one plane, while in solid geometry this is not the case. By overlooking this important distinction one may easily be led to wrong conclusions. Thus, in plane geometry two lines that do not meet are parallel (by definition of parallel lines). In solid geometry there are lines which do not meet, however far they are extended, and yet they are not parallel. Such lines are called **skew lines.** Examples of skew lines may be found by selecting certain edges of the ceiling and the floor of a room.

Before applying any theorem from plane geometry, therefore, to a three-dimensional figure, it is necessary to determine whether the elements to which the theorem is applied are in the same plane.

509. Certain of the relations of lines and planes in space are so fundamental in character that any consideration of solid figures is dependent upon them. In the introductory exercises which follow they may be taken for granted. Their logical sequence is considered in the beginning of Book VI.

1. *Two intersecting straight lines determine a plane.*

That is, they lie in a plane, and there is only one plane which contains them both.

2. *A line joining any two points in a plane lies wholly in the plane.*

3. *A line and a point not on the line determine a plane.*

4. *Three points not on the same line determine a plane.*

5. *Two parallel lines determine a plane.*

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6. *If two planes cut each other, their intersection is a straight line.*

7. *If two lines are parallel to a third line in space, they are parallel to each other* (§ 524).

The truth of all these statements is readily illustrated by very simple methods. Thus, statement 6 could be inferred from the folding of a sheet of paper.

510. The introductory exercises given in the following pages need not necessarily be completed before beginning Book VI, but may be taken along with the propositions of that book. Their main purpose is not the cultivation of logical accuracy, but rather the interpretation and visualizing of three-dimensional figures. In each case, however, there should be a definite understanding as to what is assumed in the proof.

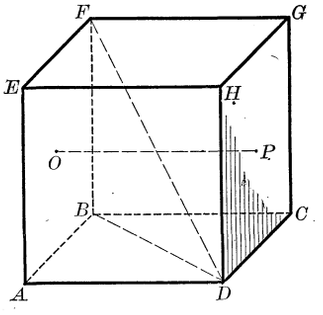
EXERCISES

A. The Cube

1. Each of the bounding squares (§ 507) of a cube is called a **face.** How many faces has a cube? How many **vertices?** How many **edges?**

2. A diagonal of any one of the bounding squares is called a *surface diagonal.* In the figure, BD is a surface diagonal. How many such diagonals has a cube?

3. A line joining two opposite vertices is called an *interior diagonal.* In the figure, DF is an interior diagonal. How many such diagonals has a cube?



4. Any angle of one of the faces is called a **face angle.** Identify the face angles ADH, DCB, ABF. How many face angles has a cube?

5. A **dihedral angle** is a figure formed by two planes meeting in a line and not extending beyond the line. Thus the planes of the squares EG and HC form a dihedral angle. It is usually denoted by writing the letters of the edge formed by the two planes between

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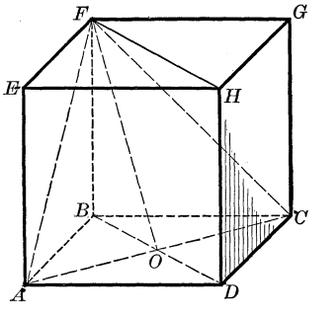
two other letters, one from each plane. Identify the following dihedral angles: *E-HD-C*, *C-BF-E*, *A-BF-G*. How many dihedral angles has a cube?

6. A line joining the centers of two opposite faces, such as *OP*, is called a *mid-line*. How many mid-lines has a cube?

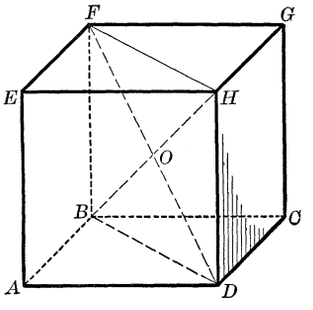
7. A **trihedral angle** is a figure formed by three rays having a common origin and not lying in a plane. Thus, in the cube the three edges meeting at *H* form the trihedral angle *H-DEG*. Identify the trihedral angles *A-BDE*, *C-BDG*. How many trihedral angles has a cube?



8. Two diagonally opposite edges of a cube, such as *BF* and *HD*, are parallel. (Why?) They determine a plane (why?) which cuts two other faces in straight lines, forming a figure called a *diagonal section*. How many diagonal sections has a cube?

9. Prove that the quadrilateral *BDHF* is a parallelogram.

10. Prove that $\square BDHF$ is a rectangle. (Draw *AC*, *AF*, *FC*. Then $\triangle ACF$ is equilateral. (Why?) Draw *FO*. Then $FO \perp AC$. (Why?) Let *a* be the length of *AB*. Determine the length of *AO*, *AF*, *FO*. Then prove that $\overline{FO}^2 = \overline{BO}^2 + \overline{BF}^2$.)



11. Prove that the diagonal sections of a cube are congruent.

12. Prove that the interior diagonals of a cube are equal.

13. Prove that the interior diagonals of a cube meet in a point which is equidistant from the vertices of the cube.

14. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within called the center. Show that all the vertices of a cube lie on the surface of a sphere. This sphere is said to be **circumscribed about the cube.**

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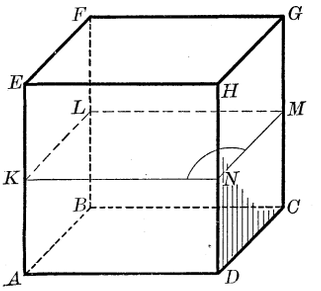
15. The quadrilateral $KLMN$, in the figure, is constructed by joining in succession the mid-points of four parallel edges of the cube. It is called a *mid-section* of the cube. How many mid-sections does a cube have? Draw the figure.



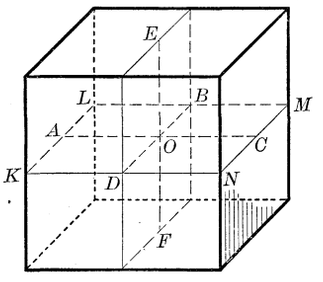
16. Prove that $KLMN$ is a square.

(If we assume the truth of § 509, KL is parallel to MN, etc.)

17. The sides of $\angle KNM$ are perpendicular to the edge DH at N. The angle KNM is called the **plane angle** of the dihedral angle A-DH-G, and is said to measure the opening of the dihedral angle, since two dihedral angles whose plane angles are equal can be proved equal by superposition. The relation between a dihedral angle and its plane angle is illustrated in opening a book or a door. If the plane angle is a right angle, the dihedral angle is a **right dihedral angle.** How many right dihedral angles has a cube?

18. Prove that the mid-sections of a cube are congruent.

19. Prove that the mid-lines of a cube are equal. (See Ex. 6. Draw two mid-sections of the cube. The mid-lines of these mid-sections are also the mid-lines of the cube.)



20. Prove that the mid-lines of a cube meet in a point.

The point O in which the mid-lines of the cube meet is the center of the sphere, which will later be defined as **inscribed** in the cube.

21. Draw all the mid-sections of a cube. How many cubes are formed?

22. Draw all the interior diagonals of a cube. The cube is now seen to consist of six figures called **pyramids.** Describe one of these pyramids.

23. If the length of the edge of a cube is e, what is the length of a surface diagonal? of an interior diagonal? What is the area of a mid-section? of a diagonal section? of the surface of the cube? of the surface of one of the pyramids described in Ex. 22?

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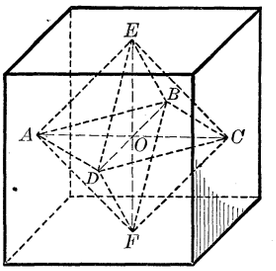
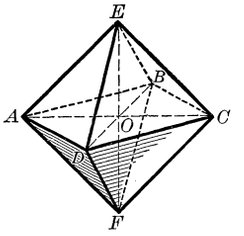
B. The Octahedron

24. Draw the mid-lines of a cube. Join the extremities of these mid-lines. These lines are the edges of a new solid, called a *regular octahedron*, that is, an 8-faced solid. How many edges has a regular octahedron? How many vertices? How many dihedral angles?



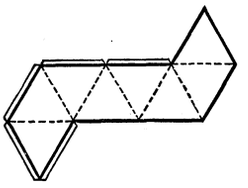
25. From the exercises on the cube, given above, it is evident that the quadrilateral $ABCD$, in the figure, is a square, and that OE equals OA. These relations make it possible to draw a regular octahedron without first drawing the circumscribed cube. If $AC = 2$ in., draw the octahedron. Take BD, in the drawing, one half its actual length, and at an angle of 45° with AC.

26. A **polyhedral angle** is a figure formed by three or more rays having a common origin, no three of the rays lying in a plane. A polyhedral angle is called *trihedral, tetrahedral*, etc., according as it is formed by three rays, four rays, etc. What sort of polyhedral angles has a regular octahedron, and how many?



27. Prove that the faces of a regular octahedron are equilateral triangles. How many degrees in each face angle?

28. The figure shows how a model of a regular octahedron may be made. The triangles are all equilateral.



29. The square $ABCD$ in the figure for Ex. 24 is called a *diagonal section* of the octahedron. How many such sections has a regular octahedron?

30. If the length of an edge of a cube is e, what is the area of one face of the inscribed regular octahedron? What is the area of a diagonal section of the octahedron?

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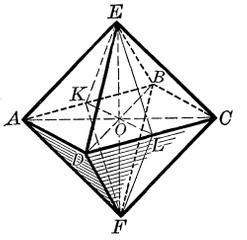
31. Show that the *diagonals* AC, BD, and EF of the regular octahedron (figure, Ex. 26) are equal, and that they meet at a point. How is this point located with reference to the vertices of the octahedron? Show that a sphere can be circumscribed about a regular octahedron (see Ex. 14).



32. If K and L are the mid-points of AB and CD respectively, $EKFL$ is a *mid-section* of the octahedron. Why is $EKFL$ a plane figure? How many mid-sections has a regular octahedron?

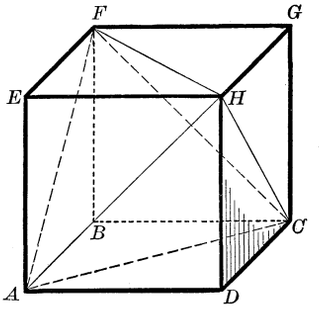
33. Prove that a mid-section of a regular octahedron is a rhombus.

34. The angle ELF is the plane angle of the dihedral angle E-CD-F. Construct the $\triangle EFL$, if $EF = 2$ in., and measure $\angle ELF$ with a protractor (or determine its value by trigonometry).

35. Find the area of a mid-section of a regular octahedron if the length of an interior diagonal, such as EF, is e.

C. The Tetrahedron

36. Draw a cube. From one of the vertices, as H, draw the three surface diagonals meeting at that point. Join the other extremities of these diagonals. These lines are the edges of a new solid $ACHF$, called a *regular tetrahedron*, or 4-faced solid. How many edges has a regular tetrahedron? How many vertices?



37. Prove that the faces of a regular tetrahedron are equilateral triangles. How many degrees in each face angle?

38. The figure (p. 335) shows how a model of a regular tetrahedron may be made. An equilateral triangle is divided into four equilateral triangles. The triangles b, c, d, can be folded about the sides of the triangle a so as to inclose a solid.

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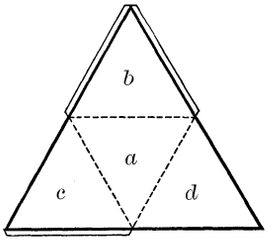
39. Find the total area of a regular tetrahedron of edge a; of the regular tetrahedron of Ex. 36 if the edge of the cube is e.

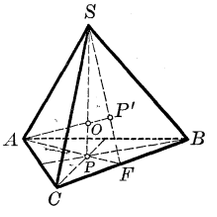
40. The **mid-section** of a regular tetrahedron is determined by one edge and the mid-point of the diagonally opposite edge. What kind of figure is it? How many mid-sections has a regular tetrahedron?



41. If the edge of a regular tetrahedron is a, find (a) the area of the mid-section, (b) the altitude on one leg of the mid-section, (c) the segments of that leg made by the altitude.

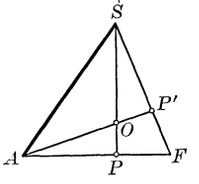
42. Through what point in the base of the tetrahedron does the altitude of the mid-section pass?

43. Show that when two mid-sections intersect, the line of intersection is an altitude.



44. Prove that two altitudes on the legs of a mid-section divide each other in the ratio of one to three. (See § 247.)

45. Show that the point of intersection of two altitudes of a mid-section is equidistant from the vertices of the tetrahedron. Find the radius of the sphere circumscribed about a regular tetrahedron of edge a.



511. A **polyhedron** is a solid bounded by planes. The intersections of the planes are called the **edges**; the polygons bounded by the edges, the **faces**; the vertices of the faces, the **vertices**; and the angles of the faces, the **face angles** of the polyhedron. The planes of any two faces meeting at an edge form a **dihedral angle**.

If a polyhedron lies entirely on the same side of the plane of every one of its faces, it is said to be **convex**; otherwise it is said to be *concave*. (*Throughout this text the word "polyhedron"* will mean a convex solid.)

512. A **regular polyhedron** is one whose faces are congruent regular polygons, the same number meeting at each vertex.

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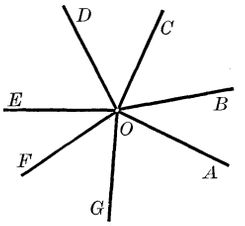
EXERCISES

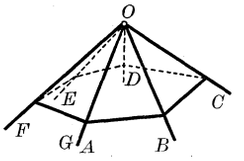
The Regular Solids

1. Show that the cube, the octahedron inscribed in the cube (Ex. 24, p. 333), and the tetrahedron inscribed in the cube (Ex. 36, p. 334) are regular solids, according to § 512.

2. It is known from plane geometry that the number of regular polygons is unlimited. The number of regular convex polyhedrons, however, is narrowly limited, as will appear presently.



Suppose that from a point O in a plane sheet of paper an indefinite number of rays are drawn, forming a set of plane angles whose sum is 360°. Imagine the paper to be folded along each one of the rays. Then, if it be desired to form from the paper a polyhedral angle about O, it will be necessary to bring two of the rays into coincidence, thus sacrificing a part of the angular magnitude about O. This simple experiment will sufficiently illustrate the important fact that *the sum of the face angles about any vertex of a convex polyhedron is less than 360°*. What is the sum of the face angles at any vertex of a cube? of a regular tetrahedron? of a regular octahedron?



3. It is now possible to set a limit to the number of possible regular convex polyhedrons. Since the bounding faces must be regular polygons, this number will depend upon the number of congruent regular polygons that can be made to meet at a vertex without violating the principle stated in Ex. 2.

If equilateral triangles be chosen as bounding surfaces, we may have three, four, or five at each vertex, but no more. (Why?) What solid has three equilateral triangles at each vertex? four?

If there are five triangles at each vertex, it will require 20 faces to complete a regular solid. It is called a **regular icosahedron.**

4. If the bounding faces are squares, we may have three at each vertex, but no more. (Why?) What regular solid can be made?

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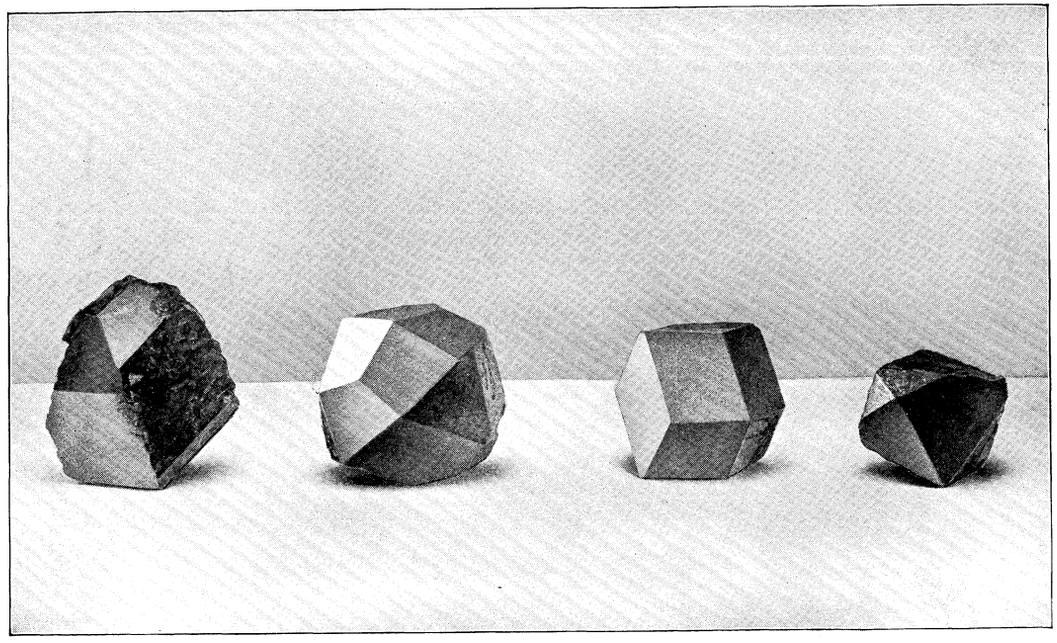

CRYSTALS

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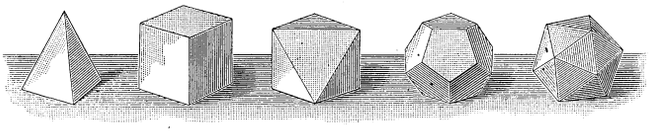
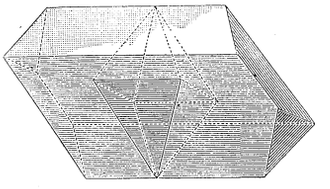
5. If the bounding faces are regular pentagons, we may have three at each vertex, but no more. (Why?) It can be shown that it will require twelve bounding surfaces to complete the solid. It is called the **regular dodecahedron.**

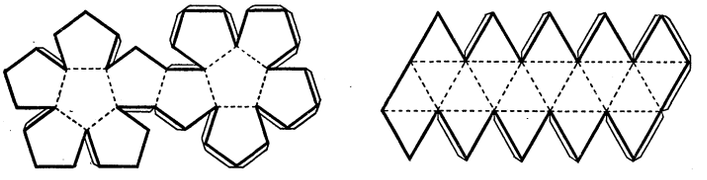
6. Show that no other regular polygons except the five mentioned above can be used to form a regular convex polyhedron.



NOTE. These five regular solids have been called the *Platonic bodies*, since their properties are known to have interested Plato and his followers, although these solids had also been studied by the Pythagoreans. The simpler ones appear in the laws of crystallography. One of these laws is to the effect that a given substance always crystallizes in forms in which the angles between corresponding faces of two crystals are the same, though the faces may not be of the same size. Thus, the regular octahedron is the basis of the crystalline form assumed by some minerals, such as magnetic iron, copper, platinum, and gold, and crystals of these substances are found in forms which can be made by planing down one or more of the faces of a regular octahedron.



The illustration on the opposite page is made from a photograph of crystals. The one on the extreme right is of the regular octahedral type.



7. The diagrams indicate how models of the regular dodecahedron and of the regular icosahedron may be constructed. Make each edge of the solids about two inches.

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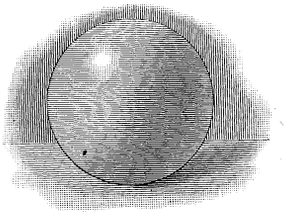
8. Complete the following table, which summarizes some of the properties of the five regular solids.

TABLE OF REGULAR SOLIDS

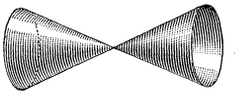
Name	Number of faces	Face	Number of faces at each vertex	Number of edges	Number of vertices	Sum of face angles
Cube	6	Square				

9. A theorem of solid geometry, which is due to the great Swiss mathematician Euler, reads as follows: In any polyhedron the number of edges is equal to the sum of the number of faces and the number of vertices, less two. Show that the solids studied thus far obey this law.

10. A theorem which results from Euler's theorem (Ex. 9) reads: The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices. Show that the polyhedrons studied thus far obey this law.



513. Curved Surfaces. All of the solids hitherto considered have been bounded by planes. But many familiar solids, such as the sphere, the cylinder, and the cone, are bounded by surfaces all of which are not plane. Such surfaces are called **curved surfaces.**



If a line-segment moves in any direction, while one of its extremities remains fixed, the other extremity lies always on a **spherical surface.** If a line revolves about an intersecting oblique line as an axis, it will be found to lie always on a **conical surface of revolution.** If a line revolves about a parallel line as an axis, it will lie on a **cylindrical surface of revolution.**

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BOOK VI

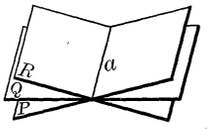
LINES AND PLANES IN SPACE

Preliminary Assumptions and Propositions

514. *A plane is unlimited in extent.*

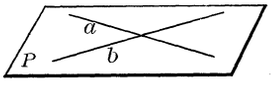
The portion of a plane under consideration is usually represented as a parallelogram (or trapezoid). A plane may be designated by a capital letter, as P, Q, etc., written within the parallelogram (or trapezoid).

515. *A line joining any two points in a plane lies wholly in the plane.*



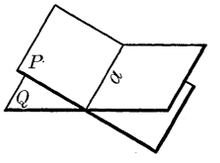
516. *Through a given line an indefinite number of planes may be passed.*

517. *Two intersecting lines determine a plane.*



518. *A line and a point not on the line determine a plane.*

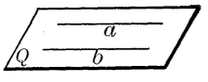
519. *Three points not on the same line determine a plane.*



520. *If two planes have one point in common, they have also another point in common.*

521. *If two planes cut each other, their intersection is a straight line.*

If they have one common point, they must have another (§ 520), and the straight line joining these two points must lie in both (§ 515). There can be no common point not on this line, for with the line it would determine one plane only (§ 518), instead of two.



522. *Two parallel lines determine a plane.*

They must lie in one plane, by the definition of parallel lines (§ 170). There can be only one plane containing both, since two such planes would intersect in two lines, which is impossible (§ 521).

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EXERCISES

1. How does a mason test the flatness of a plastered surface?

2. Why is a three-legged stool always steady?

3. Lines are drawn from a given point to points on a given line. What can be asserted of all these lines?

4. How may three points be placed so as not to determine a plane?

5. What plane figures are determined by three points that determine a plane?

6. If three lines pass through the same point, but do not lie in the same plane, how many planes do they determine? four such lines? five? n?

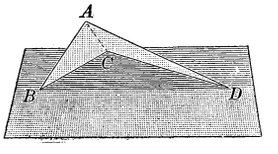
7. How many lines are determined by three mutually intersecting planes which do not pass through the same line? by four such planes, no three of which pass through the same line? five? n?

8. How many planes are determined by four points not in the same plane, no three of which lie on the same line? by five, no four of which lie in the same plane, and no three on the same line?

9. The formula for the number of planes determined by n points under the conditions of Ex. 8 is $\frac{1}{6}\,n\,(n-1)\,(n-2)$. Test the answers to Ex. 8 by this formula, and find the number of planes determined by six such points; seven.

10. Two-line segments are skew to each other (§ 508). How many planes are determined by their end-points? How many lines? What solid figure?

11. A "skew quadrilateral" consists of four lines which join in succession four points not all in the same plane. If a card $ABCD$ is partially folded along a diagonal, the edges form a skew quadrilateral. In the figure, $ABCD$ is a skew quadrilateral, and AC and BD are its diagonals. Show that the diagonals of a skew quadrilateral cannot intersect.



12. The path of the earth around the sun is elliptical, and that of the moon around the earth is elliptical. Are the planes of the two orbits coincident? Are they intersecting or parallel? Give a reason for your answer based on the frequency of total eclipses.

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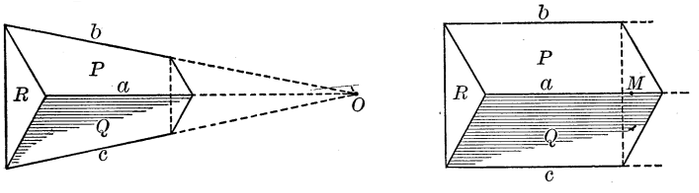
PROPOSITION I. THEOREM

523. *Three mutually intersecting planes not passing through the same line determine three lines, which either meet in a point or are parallel each to each.*



Given the three mutually intersecting planes *P*, *Q*, and *R*.

To prove *that P, Q, and R determine three lines which either meet in a point or are parallel.*

Proof. 1. Let P and Q meet R in the lines b and c respectively, and let P and Q meet each other in the line a.

Then b and c either meet in a point, or are parallel. § 170

2. Suppose that b and c meet in the point O.

Then O must lie in P, and also in Q.

3. Hence O must lie on a, the intersection of P and Q. § 521

That is, a, b, and c meet in a point.

4. Suppose $b \parallel c$. Then a cannot meet b, for if it did, c would meet b in the same point.

5. But a and b lie in P, and hence $a \parallel b$.

6. In like manner $a \parallel c$.

That is, a, b, and c are parallel.

524. COROLLARY. *Two lines parallel to a third line are parallel to each other.*

Given $a \parallel$ to b, and $c \parallel$ to b; to prove $a \parallel$ to c.

Since $a \parallel b$, they determine a plane P; since $c \parallel b$, they determine a plane R. Let M be a point in a, and let M and c determine the plane Q. Then Q meets P in a line d, which is $\parallel$ to b and also to c (§ 523). This line must coincide with a, since a and d are both $\parallel$ to b, and pass through M (Parallel Axiom, § 174). Hence $a \parallel c$.

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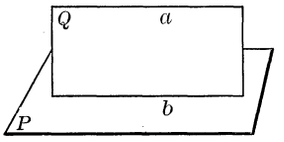
PROPOSITION II. THEOREM

525. *If two lines are parallel, any plane containing one of these lines and only one cannot meet the other line.*



Given two parallel lines *a* and *b*, and the plane *P* containing *b*, but not *a*.

To prove *that* *P does not meet a.*

Proof. 1. Let a and b determine a plane Q.

2. If a could meet P, their point of intersection would lie in Q. This point would therefore lie in b, the intersection of P and Q.

3. But this is impossible.

(Since $a \parallel b$ by hypothesis.)

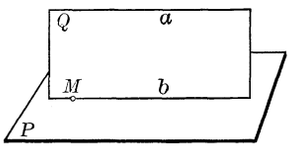
Hence P does not meet a.

526. A line is **parallel to a plane** when the line and the plane do not meet, no matter how far they are produced.

To draw through a given point a line parallel to a given plane, draw it parallel to any line in the plane (§ 173).

To pass through a given point a plane parallel to a given line, pass it through a line which is parallel to the given line.

527. COROLLARY 1. *If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane through the line.*



If $a \parallel P$, and if Q, a plane passed through a, intersects P in b, then a cannot meet b, otherwise it would meet P. But a and b lie in Q. Hence $a \parallel b$, since they lie in the same plane and do not meet.

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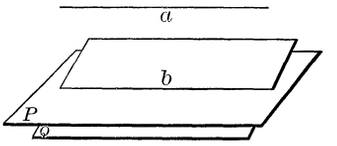
528. COROLLARY 2. *If a line is parallel to the intersection of two planes, it is parallel to each of the planes.*

529. COROLLARY 3. *If a line is parallel to a plane, the line which passes through a point in the plane and is parallel to the given line lies in the plane.*

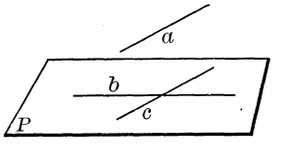


Since this line must coincide with the intersection of the given plane with the plane determined by the given line and the point (Parallel Axiom). (See figure of § 527.)

530. COROLLARY 4. *A line parallel to each of two intersecting planes is parallel to their intersection.*

Since a line through a point in the intersection, and parallel to the given line, must lie in both planes (§ 529), and must therefore be their intersection.

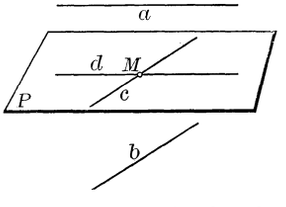
531. COROLLARY 5. *Through a given line one plane can be passed parallel to a given skew line, and only one.*



If a is skew to b, through any point in b draw $c \parallel$ to a. Then b and c determine a plane P (§ 517), which is parallel to a (§ 525).

There is no other plane through b parallel to a, for if there were such a plane, the intersection b of the two planes would be parallel to a (§ 530). But b is skew to a. So there is no second plane through b parallel to a.

532. COROLLARY 6. *Through a given point one plane can be passed parallel to each of two given skew lines, and only one.*



If a is skew to b, through the given point M draw $d \parallel$ to a, and $c \parallel$ to b. Then d and c determine a plane P, which is parallel to a and b. (Why?)

If there were two such planes, they would intersect in a line which would be parallel to a (why?) and also to b. But a and b would then be parallel (why?), which is impossible, since they are skew.

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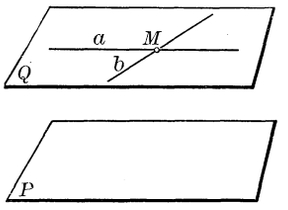
PROPOSITION III. THEOREM

533. *If two intersecting lines are each parallel to a plane, the plane of those lines will not intersect the given plane.*



Given the intersecting lines *a* and *b*, each parallel to the plane *P*; and *Q*, the plane of *a* and *b*.

To prove *that* Q *does not intersect* P.

Proof. 1. If Q did intersect P, the intersection would be a line (§ 521) which would be parallel to a (§ 527) and in like manner also to b.

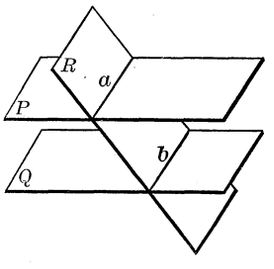
2. But this is impossible (Parallel Axiom).

Hence Q does not meet P.

534. Two planes are parallel when they do not intersect, no matter how far they are produced.

535. COROLLARY 1. *Through a given point to pass a plane parallel to a given plane.*

Draw two intersecting lines through the point, each parallel to the given plane (§ 526).



536. COROLLARY 2. *The intersections of two parallel planes by a third plane are parallel lines.*

The intersections a and b cannot meet, for otherwise P and Q would meet. But a and b lie in R. Hence $a \parallel b$.

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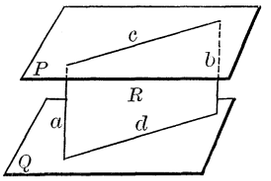
537. COROLLARY 3. *Through a given point only one plane can be passed parallel to a given plane.*

If there were two such planes, then a fourth plane could be passed through the given point, cutting the other three planes so as to determine three lines of intersection, two of which, passing through the given point, would be parallel to the third (§ 536), which is impossible.



538. COROLLARY 4. *Parallel lines included between parallel planes are equal.*

The lines determine a third plane, which intersects the given planes in parallel lines (§ 536), forming a parallelogram (§ 196) whose opposite sides are equal (§ 201).

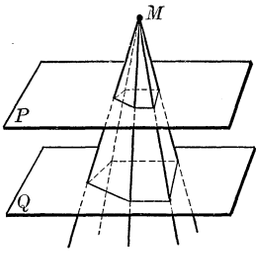
EXERCISES

1. What is the shape of the section made by a slanting cut across a plank?

2. What is the position of a pencil relative to the table when it is parallel to its shadow on the table?

3. Each edge of a cube is parallel to how many faces? to how many diagonal sections? to how many edges? (See Ex. 8, p. 331.)

4. Three planes are each parallel to a given line. Describe their relative positions (a) if they pass through the same point; (b) if they do not.



5. Lines are drawn from a given point, M, so as to intersect two parallel planes, P and Q. These lines determine planes. What can be said of the intersections of these planes with the planes P and Q? What can be said of the segments of the lines? (§ 372.)

6. Prove that if two planes are each parallel to the same plane, they are parallel to each other.

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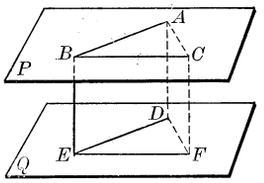
PROPOSITION IV. THEOREM

539. *If two angles, not in the same plane, have their sides respectively parallel and lying on the same side of the line joining their vertices, they are equal, and their planes are parallel.*



Given the angles *ABC* and *DEF* in the planes *P* and *Q* respectively, with *BA* parallel to *ED*, and *BC* parallel to *EF*, each pair of rays lying on the same side of the line *BE*.

To prove *that* $\angle ABC = \angle DEF$, *and that* $P \parallel Q$.

Proof. 1. Take $BA = ED$, and $BC = EF$.

Draw CF, AD, AC, and DF.

2. Then AE is a parallelogram, Why?

and BF is a parallelogram. Why?

3. Hence AD is equal and parallel to BE,

and CF is equal and parallel to BE.

Therefore AD is equal and parallel to CF. Ax. 1 and § 524

4. Hence $ADFC$ is a parallelogram, Why?

and $AC = DF$.

5. $\therefore \triangle ABC \equiv \triangle DEF$, Why?

and $\angle ABC = \angle DEF$.

6. Also $AB \parallel Q$, and $BC \parallel Q$. § 525

$\therefore P \parallel Q$. § 533

NOTE. It is tacitly assumed that the five triangle congruences (§§ 133, 134, 142, 165, 166) of plane geometry are true when the triangles are in *different* planes. This follows from the **axiom of superposition** (p. 74, Plane Geometry, Remark), which is not limited to a single plane in its application.

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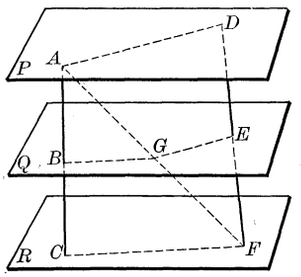
PROPOSITION V. THEOREM

540. *If two lines are cut by three parallel planes, their corresponding segments are proportional.*



Given the parallel planes *P, Q,* and *R,* intersecting two straight lines at *A, B, C,* and *D, E, F,* respectively.

To prove *that* $AB : BC = DE : EF$.

Proof. 1. Draw AF, and pass a plane through AC and AF, intersecting the planes Q and R in BG and CF respectively.

2. Then $CF \parallel BG$. Why?

3. In like manner, if a plane is passed through AF and DF,
$GE \parallel AD$.

4. Hence $AB : BC = AG : GF$, Why?

and $DE : EF = AG : GF$. Why?

Hence $AB : BC = DE : EF$. Ax. 1

541. COROLLARY 1. *If two straight lines are cut by four or more parallel planes, the corresponding segments of the lines are proportional.*

542. COROLLARY 2. *If two intersecting lines are cut by two parallel planes neither of which passes through their point of intersection, the corresponding segments are proportional.*

(A third parallel plane may be passed through the point of intersection.)

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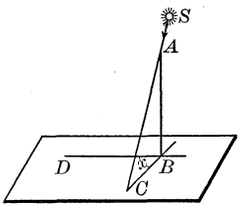
543. Projection. The picture or outline of a three-dimensional figure on a plane is called a **projection** of this figure. Rays from a common origin are supposed to pass through the determining points of the solid and then to be intercepted by a plane, thus producing the picture. The plane is called the **plane of projection** (or *screen*), the rays are called **projecting rays,** and their origin is called the **center of projection.**

544. If the center of projection is moved very far away, the rays become very nearly parallel. When the rays are parallel it is customary to speak of **parallel projection,** as distinguished from **central projection.**

A shadow produced by the rays of the sun will serve to illustrate parallel projection, while a shadow produced by an accessible source of light, such as an arc light, furnishes an example of central projection.

In section 545 and the exercises following we shall consider parallel projection only. The projecting rays are supposed not to be parallel to the screen.

545. Principles of Parallel Projection. The projection of a solid depends not only on the form of the solid, but also on the position of the plane on which the projection is to be taken, and on the direction of the projecting rays. It is therefore desirable to establish certain conventions in regard to the direction of the projecting rays.



In the figure let AB represent a vertical rod and BC its shadow on a horizontal plane or screen. If BD is a fixed horizontal line, the angle CBD ($\angle x$) may assume an indefinite number of values, and the length of BC may vary indefinitely. For practical reasons the $\angle x$ is usually taken as 30°, 45°, or 60°, and the ratio of BC to AB is taken as 1:3, 1:2, and 2:3 respectively. Thus, if $\angle x$ is 45°, make BC equal to $\frac{1}{2}\,AB$. These values, "45° and half-length," are characteristic of "cabinet projection" (§ 507). On page 349 it is proved that BC is a straight line.

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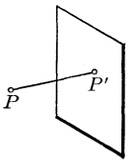
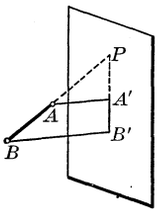
The following principles of parallel projection are seen to follow directly from the preceding propositions:

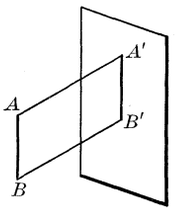
The projection of a point is the point in which its projecting ray intersects the screen.

The projection of a straight line (or a line-segment) is a straight line (or a line-segment), unless the line is parallel to the projecting rays, in which case the projection is a point.



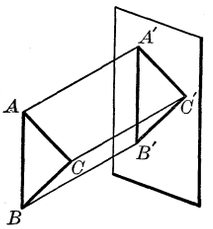
Suggestion. The projection of the line AB is the locus of the projections of its points (§ 543). The projecting ray of any point on AB will lie in the plane of the projecting rays of A and B (Parallel Axiom).

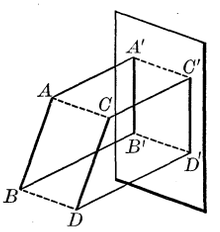




If a line-segment is parallel to the screen, its projection is equal and parallel to it. (Why?)

The projection of a triangle whose plane is parallel to the screen is a congruent triangle. (Why?)





The projection of a plane figure whose plane is parallel to the screen is a figure congruent to the original (§ 144).

The projections of parallel lines are also parallel. (Why?)

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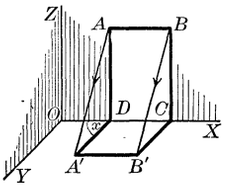
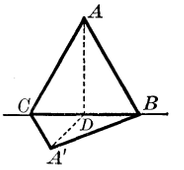
EXERCISES

1. In the figure let OXZ represent a wall of the recitation room, and let OXY represent the floor. Let $ABCD$ be the outline of a door of the room. Imagine the rays of the sun to outline on the floor a picture of this opening, namely, $A'B'CD$. Take $\angle x$ as 45°, and locate A' and B' by taking $A'D$ equal to $\frac{1}{2}\ AD$, and $B'C$ equal to $\frac{1}{2}\ BC$.



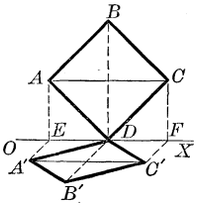
2. Draw a parallel projection on a horizontal plane of an equilateral triangle which is situated in a vertical plane, one of its sides lying in the horizontal plane.

(Draw the altitude AD and determine A' as in Ex. 1. Then $\triangle A'BC$ is the required projection.)

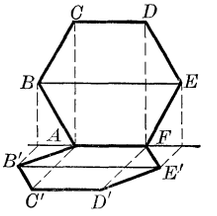


3. Draw a projection on a horizontal plane of a square which is situated in a vertical plane, one of its vertices lying in the horizontal plane and one of the diagonals being vertical.

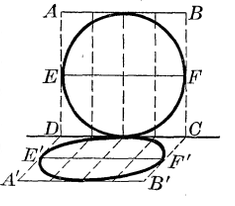
(Draw AE and CF perpendicular to OX and locate A', B', C', in the usual way. Observe that $A'C'$ is equal and parallel to AC. Why?)



4. Draw a projection on a horizontal plane of a regular hexagon, one of its sides lying in the horizontal plane.



5. The last figure on this page shows how a projection on a horizontal plane of a circle may be obtained approximately, one diameter being perpendicular to the horizontal plane. Circumscribe a square about the circle. Divide the diameter EF into a convenient number of equal parts. Through the points of division draw lines parallel to AD and BC. The intersections of these parallels with the circumference can easily be projected. In this way the projection of the circle may be determined approximately. This projection of a circle is called an *ellipse*.



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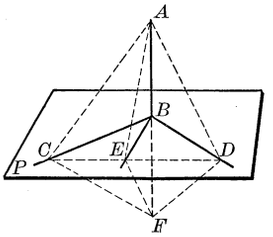
PERPENDICULAR LINES AND PLANES

Proposition VI. Theorem

546. *If a straight line is perpendicular to each of two straight lines in a plane, at their point of intersection, it is perpendicular to any line lying in the plane and passing through the point of intersection.*



Given AB perpendicular to BC and BD in the plane P, and BE any other line through B and lying in P.

To prove *that* $AB \perp BE$.

Proof. (Outline) 1. Draw CD, cutting BC, BE, and BD in C, E, and D, respectively. Produce AB its own length to F. Draw AC, AD, AE, FC, FD, FE.

2. Then $\triangle ABC \equiv \triangle FBC$ (why?) and $AC = FC$.
3. Also $\triangle ABD \equiv \triangle FBD$ (why?) and $AD = FD$.
4. Hence $\triangle ACD \equiv \triangle FCD$ (why?) and $\angle ACE = \angle FCE$.
5. Then $\triangle ACE \equiv \triangle FCE$ (why?) and $AE = FE$.
6. Hence $\triangle ABE \equiv \triangle FBE$ (why?) and $\angle ABE = \angle FBE$.

$\therefore AB \perp BE$. Why?

547. A line is **perpendicular to a plane** when it is perpendicular to all lines in the plane drawn through its foot. In that case, also, the plane is said to be **perpendicular to the line.**

548. A line is **oblique to a plane** when it intersects the plane but is not perpendicular to it.

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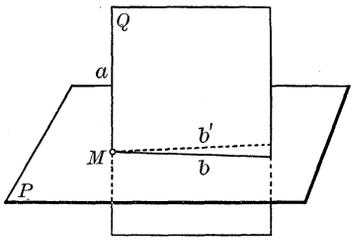
PROPOSITION VII. THEOREM

549. *All the perpendiculars that can be drawn to a given line at a given point lie in a plane perpendicular to the line at the given point.*



Given *b*, any line perpendicular to the line *a* at the point *M*, and the plane *P* also perpendicular to *a* at *M*.

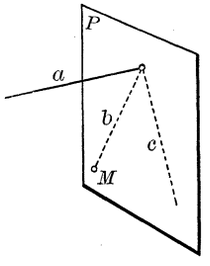
To prove *that* *b lies in P.*

Proof. 1. Pass a plane Q through a and b; then Q intersects P in a line b'.

2. Then	$a \perp b'$.	Def.
3. But	$a \perp b$.	Hyp.
4.	$\therefore b$ coincides with b'	Why?
5. Hence	b lies in P.	

In like manner any other line $\perp$ to a at M lies in P.

550. COROLLARY. *Through a given point one and only one plane can be passed perpendicular to a given line.*



If the given point M is without the given line a, from M draw $b \perp$ to a. At the foot of b, but not in the plane of a and b, draw $c \perp$ to a; then b and c determine the required plane P. If M is on a, two $\perp$s to a at M determine the required plane.

If there were two such planes, a third plane could be passed through the given line, cutting these two planes so as to determine two lines $\perp$ to the same line in the same plane, at, or from, the same point, which is impossible.

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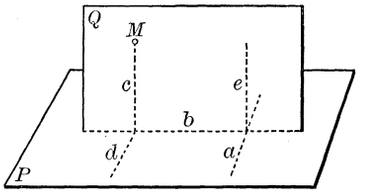
PROPOSITION VIII. THEOREM

551. *Through a given point one and only one line can be drawn perpendicular to a given plane.*



Given the plane P and the point M.

To prove *that one and only one line can be drawn through M perpendicular to P.*

Proof. CASE I. *When M is not in P.*

1. In P draw a line a. Through M pass a plane $Q \perp$ to a (§ 550), intersecting P in b. In Q draw $c \perp$ to b through M. In P, at the foot of c, draw $d \parallel$ to a, and in Q, at its intersection with a, draw $e \parallel$ to c.

2. Then	$\angle cd = \angle ae$.	§ 539
3. But	$\angle ae$ is a right angle.	§ 547
	$\therefore \angle cd$ is a right angle.	
4. Also	$c \perp b$.	Cons.
	$\therefore c \perp P$.	§ 546

5. If there were two such $\perp$s, they would both be $\perp$ to the intersection of their plane with P, which is impossible.

CASE II. *When M is in P.*

Proof. In P draw a line a not passing through M.

(Complete as in Case I.)

552. COROLLARY. *The perpendicular is the shortest line from a point to a plane.*

553. The **distance of a point from a plane** is the length of the perpendicular from the point to the plane.

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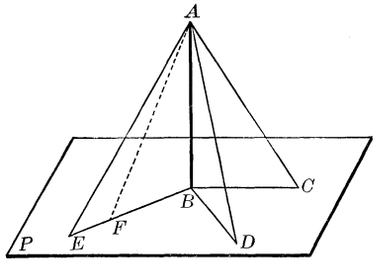
PROPOSITION IX. THEOREM

554. *Oblique lines drawn from a point in a perpendicular to a plane and cutting off equal distances from the foot of the perpendicular are equal; and of two such lines cutting off unequal distances the one cutting off the greater distance is the greater; and conversely, equal oblique lines cut off equal distances from the foot of the perpendicular, and of two unequal oblique lines the greater line cuts off the greater distance.*



(a) **Given *AB* perpendicular to the plane *P* at *B*, the lines *BC* and *BD* equal, and *BE* greater than *BC*.**

To prove *that* $AC = AD$, *and* $AE > AC$.

Proof. (Outline). 1. $\triangle ABC \equiv \triangle ABD$. Why?

$\therefore AC = AD$.

2. Take $BF = BC$. Draw AF.

Then $AE > AF$. § 233

But $AF = AC$. $\therefore AE > AC$.

(b) **Given *AB* perpendicular to the plane *P* at *B*, *AC* and *AD* equal, and *AE* greater than *AC*.**

To prove *that* $BC = BD$, *and* $BE > BC$.

Proof. (Outline). 1. $\triangle ABC \equiv \triangle ABD$. $\therefore BC = BD$. Why?

2. BE is neither equal to BC nor less than BC. Why?

$\therefore BE > BC$.

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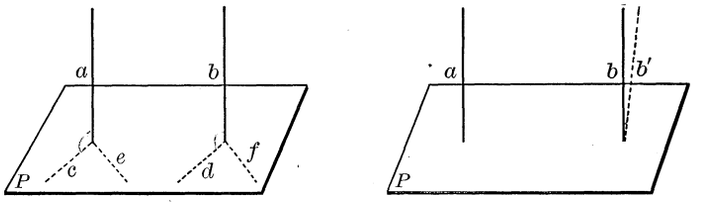
PROPOSITION X. THEOREM

555. *If one of two parallel lines is perpendicular to a plane, the other is also; and conversely, two lines perpendicular to the same plane are parallel.*



(a) **Given the parallel lines *a* and *b*, with the line *a* perpendicular to the plane *P*.**

To prove *that* $b \perp P$.

Proof. 1. Through the foot of a draw c in P; and through the foot of b draw $d \parallel$ to c.

2. Then	$\angle ac = \angle bd$.	§539
3. But	$a \perp c$.	Hyp.
4.	$\therefore b \perp d$.	

5. In like manner, we may prove b perpendicular to f, another line in P.

$\therefore b \perp P$. Why?

(b) **Given the lines *a* and *b*, each $\perp$ to the plane *P*.**

To prove *that* $a \parallel b$.

Proof. 1. Through the foot of b draw $b' \parallel$ to a.

2. Then	$b' \perp P$.	Why?
3. But	$b \perp P$.	Hyp.
4.	$\therefore b$ coincides with b',	Why?

and $a \parallel b$.

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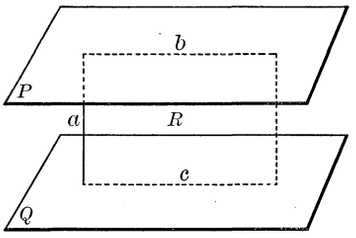
PROPOSITION XI. THEOREM

556. *A straight line perpendicular to one of two parallel planes is perpendicular to the other; and conversely, two planes perpendicular to the same line are parallel.*



(a) **Given the line *a* perpendicular to the plane *P*, and *P* parallel to the plane *Q*.**

To prove *that* $a \perp Q$.

Proof. 1. Through a pass the plane R, intersecting P in the line b and Q in the line c.

2. Then $b \parallel c$. Why?

3. But $a \perp b$. Hyp.

$\therefore a \perp c$. Why?

4. In like manner, by passing another plane S through a, intersecting P in the line d and Q in the line e, it can be proved that a is perpendicular to another line e in Q.

$\therefore a \perp Q$. Why?

(b) **Given the planes *P* and *Q* each perpendicular to the line *a*.**

To prove *that* $P \parallel Q$.

Proof. P and Q cannot meet, for if they did, they would be perpendicular to the same line from the same point. But this is impossible (§ 550).

$\therefore P \parallel Q$.

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557. The **distance between two parallel planes** is the length of the segment of a common perpendicular cut off between the two planes.

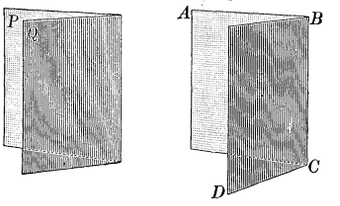
558. COROLLARY. *Two parallel planes are everywhere equally distant* (§ 538).

559. Measure of a Dihedral Angle. A dihedral angle (p. 330) is a figure formed by two planes meeting in a line, called the **edge** of the dihedral angle, and not extending beyond the line.

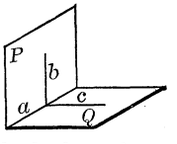
The planes forming a dihedral angle are called the **faces** of the dihedral angle.

The magnitude of a dihedral angle is determined in the same manner as that of a plane angle, namely, by the amount of revolution of a plane about the edge of the angle from one face of the angle to the other. A complete revolution will bring the plane back to its starting point, and this amount of angular magnitude may be taken as a standard with which other dihedral angles may be compared.

NOTATION. A dihedral angle is designated by the planes which form its faces, as, the dihedral angle PQ; or by the line which forms its edge, and one point in each of its two faces, as, the dihedral angle A-BC-D.



560. Adjacent dihedral angles, equal dihedral angles, right dihedral angles, vertical dihedral angles, alternate-interior dihedral angles, etc., may be defined by conditions similar to those for the corresponding cases in plane angles. (See Plane Geometry, §§ 25, 26, 40, 168.)



561. The **plane angle** of a dihedral angle is the angle formed by two lines, one in each face, each perpendicular to the edge of the dihedral angle at the same point.

Any two plane angles of a dihedral angle are equal (§ 539).

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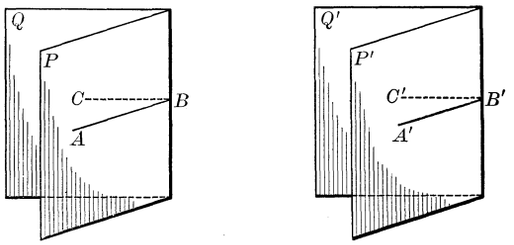
PROPOSITION XII. THEOREM

562. *Two dihedral angles are equal if their plane angles are equal; and conversely, equal dihedral angles have equal plane angles.*



(a) **Given that angle *ABC* is the plane angle of the dihedral angle *PQ*, and that angle *A'B'C'* is the plane angle of the dihedral angle *P'Q'*; and that the angles *ABC* and *A'B'C'* are equal.**

To prove *that* $\text{dihedral } \angle PQ = \text{dihedral } \angle P'Q'$.

Proof. 1. Place the dihedral $\angle PQ$ upon the dihedral $\angle P'Q'$ so that $\angle ABC$ and $\angle A'B'C'$ coincide.

2. Then the edges of the dihedral angles will coincide. Why?

3. $\therefore$ P coincides with P', and Q with Q'. Why?

$\therefore$ dihedral $\angle PQ =$ dihedral $\angle P'Q'$.

(b) **Given that angle *ABC* is the plane angle of the dihedral angle *PQ*, and that angle *A'B'C'* is the plane angle of the dihedral angle *P'Q'*; and that the dihedral angles *PQ* and *P'Q'* are equal.**

To prove *that* $\angle ABC = \angle A'B'C'$.

Proof. 1. Place the dihedral $\angle PQ$ upon the dihedral $\angle P'Q'$ so that they coincide and point B falls on point B'.

2. Then AB coincides with $A'B'$, and CB with $C'B'$. Why?

$\therefore \angle ABC = \angle A'B'C'$.

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563. Taking any multiple of a dihedral angle involves taking an equimultiple of its plane angle; and taking any fractional part of a dihedral angle involves taking the same fractional part of its plane angle; therefore,

A dihedral angle is measured by its plane angle.

For example, a dihedral angle whose plane angle is a right angle is a right dihedral angle, etc. Hence the plane angle may be called the **measuring angle** of the dihedral angle.

564. Perpendicular Planes. Each of the planes forming a right dihedral angle is said to be **perpendicular** to the other plane.

565. Many properties of dihedral angles are established by methods precisely similar to those employed in the cases of the corresponding properties of plane angles. The theorems which arise may for the most part be accepted without formal demonstration, or they may be treated as exercises. It should be noted, however, that to the *parallel axiom* of plane geometry there corresponds a *theorem* (§ 537) relating to planes, which is not assumed but proved.

Some of these theorems are as follows:

(a) *When one plane intersects another, the adjacent dihedral angles are supplementary, and the vertical dihedral angles are equal.*

(b) *If two parallel planes are cut by a third plane, the alternate-interior dihedral angles are equal; the alternate-exterior dihedral angles are equal; the interior dihedral angles on the same side of the cutting plane are supplementary, and the exterior dihedral angles on the same side are supplementary.*

(c) *Two dihedral angles whose faces are respectively parallel are either equal or supplementary.*

(d) *A plane perpendicular to one of two parallel planes is perpendicular to the other.*

(e) *Two planes perpendicular to the same plane are parallel if their lines of intersection with that plane are parallel.*

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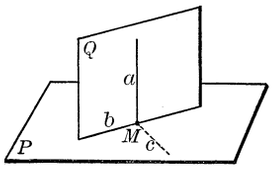
PROPOSITION XIII. THEOREM

566. *If a line is perpendicular to a plane, every plane passed through the line is perpendicular to the given plane.*



Given the line a perpendicular to the plane P at M, and Q any plane containing a.

To prove *that* $Q \perp P$.

Proof. 1. In the plane P, through M, the foot of a, draw $c \perp$ to b, the intersection of P and Q.

2. Then $a \perp b$, Why?

and $\angle ac$ is the measure of the dihedral $\angle PQ$. § 563

3. Hence $a \perp c$. Why?

$\therefore Q \perp P$. Why?

567. COROLLARY 1. *Through a given point to pass a plane perpendicular to a given plane.*

Through the point draw a line $\perp$ to the given plane (§ 551). Then any plane through this line is perpendicular to the given plane.

568. COROLLARY 2. *A plane perpendicular to the edge of a dihedral angle is perpendicular to the faces.*

EXERCISES

1. If three lines from a common point are mutually perpendicular, what can be proved regarding the planes determined by the lines?

2. Prove that a line and a plane which are both perpendicular to the same line are parallel.

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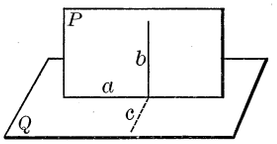
PROPOSITION XIV. THEOREM

569. *If two planes are perpendicular to each other, a line in one of them perpendicular to their intersection is perpendicular to the other.*



Given the plane *P* perpendicular to the plane *Q*, intersecting *Q* in the line *a*; and the line *b* in *P* perpendicular to *a*.

To prove *that* $b \perp Q$.

Proof. 1. In Q, through the foot of b, draw $c \perp$ to a.

2. Then $\angle bc$ is a right angle.

(Being the measure of a right dihedral angle.)

That is, $b \perp c$.

3. But $b \perp a$. Hyp.

$\therefore b \perp Q$. Why?

570. COROLLARY 1. *If two planes are perpendicular to each other, a line perpendicular to one of them at any point in their intersection lies in the other.*

(Prove by the indirect method.)

571. COROLLARY 2. *If two planes are perpendicular to each other, a line perpendicular to one of them from any point in the other lies in the other.*

EXERCISES

1. Are two lines perpendicular to the same line in space necessarily parallel?

2. Show how, with a ten-foot pole, compasses, and a ruler, to find a point on the floor directly beneath a point on a nine-foot ceiling.

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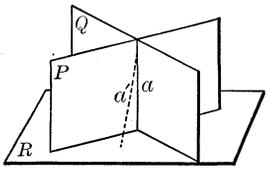
PROPOSITION XV. THEOREM

572. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to that plane.*



Given the planes P and Q, each perpendicular to the plane R, and intersecting in the line a.

To prove *that* $a \perp R$.

Proof. 1. From any point in a draw $a' \perp$ to the plane R.

2. Then a' lies in P, and also in Q. § 571

3. $\therefore$ a' is the intersection of P and Q.

4. Hence a' coincides with a, and $a \perp R$.

573. The foregoing theorem may be stated: *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection; that is, a plane perpendicular to the faces of a dihedral angle is perpendicular to the edge.*

EXERCISES

1. Three pairs of parallel planes mutually intersect so as to enclose a space. What may be said of the lines thus determined? of the plane figures thus determined? Prove your answers.

2. If a right angle is rotated about one side as an axis, what does the other side generate?

3. If three or more planes intersect in parallel lines, what may be said of a plane perpendicular to one of the lines? Prove your answer.

4. How does a carpenter determine the number of degrees between two adjoining walls of a room by the use of a bevel and a protractor?

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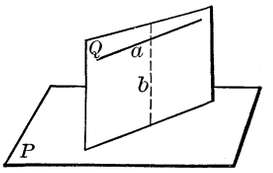
PROPOSITION XVI. THEOREM

574. *Through a given line oblique or parallel to a plane one and only one plane can be passed perpendicular to the given plane.*



Given the line *a* oblique or parallel to the plane *P*.

To prove *that one plane and only one can be passed through the line a perpendicular to P.*

Proof. 1. From any point in a without the plane P draw a line $b \perp$ to P.

2. Then a and b determine a plane Q.

3. $\therefore\ Q \perp P$. § 566

4. If there were two such planes, Q and Q', they would intersect in a, which would then be $\perp$ to P. § 572

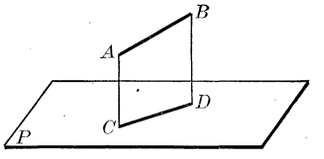
But a is oblique or parallel to P. Hyp.

Hence there is only one such plane.

575. The **orthogonal projection of a point on a plane** is the foot of a perpendicular let fall on the plane from the point.

576. The **orthogonal projection on a plane of an oblique or a parallel line** is the intersection of the plane with a *perpendicular* plane passed through the line. Evidently this projection is a straight line. (§ 521.)

Thus CD is the orthogonal projection of AB on the plane P.



The orthogonal projection of a line-segment is the locus of the projections of its points. (§ 571.)

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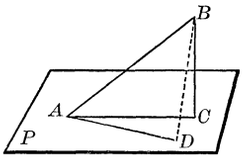
PROPOSITION XVII. THEOREM

577. *The acute angle formed by a line and its orthogonal projection upon a plane is the least angle which the line makes with any line in the plane.*



Given AC, **the projection of the line** AB **upon the plane** P, **and** AD **any other line drawn through** A **in** P.

To prove *that* $\angle BAC < \angle BAD$.

Proof. 1. Take $AD = AC$. Draw BD.

2. In the $\triangle ABC$ and ABD

AB is common,

$AC = AD$,

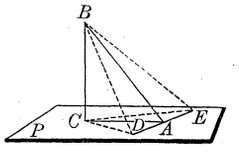
and $BC < BD$. Why?

3. $\therefore \angle BAC < \angle BAD$. § 232

578. The **angle between a line and a plane** is the angle between the line (produced if necessary) and its orthogonal projection on the plane.

EXERCISES

1. Theorem. If a line meets its orthogonal projection on a plane, a line in the plane perpendicular to one of them at their point of intersection is perpendicular to the other also.



Suggestion. From the point of intersection of AB and its projection AC, lay off $AD = AE$ on DE which is given $\perp$ to AB, and compare $\triangle$.

2. A line ten inches long makes an angle of 30° (45°, 60°, 90°), with a plane. Find the length of the orthogonal projection.

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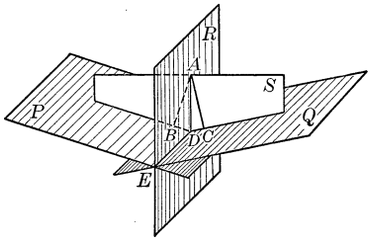
PROPOSITION XVIII. THEOREM

579. *Every point in a plane bisecting a dihedral angle is equidistant from the faces of the angle; and conversely, every point equidistant from the faces of a dihedral angle lies in the plane that bisects it.*



(a) **Given *A*, any point in the plane *R* which bisects the dihedral angle *PQ*.**

To prove *that* A *is equidistant from* P *and* Q.

Proof. 1. From A let fall AB and $AC \perp$ to P and Q respectively (§ 551). Let AB and AC determine the plane S, which intersects P, Q, and R in DB, DC, and DA respectively. Let ED be the edge of the dihedral angle PQ.

2. Then $S \perp P$ and $S \perp Q$. Why?

3. $\therefore S \perp DE$. Why?

4. Hence DE is perpendicular to DB, DC, and DA.

That is, $\angle BDA$ is the measure of the dihedral angle PR,

and $\angle CDA$ is the measure of the dihedral angle QR.

5. Also $AB \perp BD$ and $AC \perp CD$. Why?

6. Then $\triangle ABD \equiv \triangle ACD$. Why?

From which $AB = AC$.

That is, A is equidistant from P and Q.

(Converse proof to be completed.)

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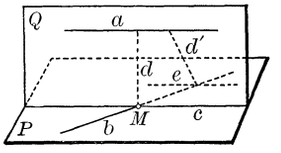
PROPOSITION XIX. THEOREM

580. *Between two skew lines one and only one common perpendicular can be drawn.*



Given the skew lines a and b.

To prove *that one and only one common perpendicular can be drawn between a and b.*

Proof. 1. Through b pass a plane $P \parallel$ to a (§ 531). Through a pass a plane $Q \perp$ to P (§ 574), intersecting P in c. Lines b and c must intersect. (Why?) Call their point of intersection M. In Q and through M draw $d \perp$ to c.

2. Then	$a \parallel c$,	Why?
and	$d \perp a$.	Why?
3. Also	$d \perp P$.	Why?
4.	$\therefore d \perp b$.	Why?

That is, d is the common perpendicular required.

5. Suppose there were another common perpendicular d'. Then in P and through the point of intersection of d' and b draw $e \parallel$ to a.

6. Then	d' would be $\perp$ to e,	Why?
and	d' would be $\perp$ to P.	Why?
7. Hence	d' would lie in Q,	Why?
and	d' would meet b at M.	Why?
8. Hence	d' would coincide with d.	Why?
That is,	d is the only common perpendicular.	

581. COROLLARY. *The common perpendicular is the shortest line that can be drawn between two skew lines.*

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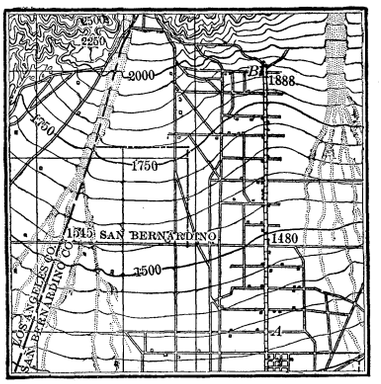
REVIEW EXERCISES

1. If P, Q, and R are planes, a, b, and c, lines, and K and L, points, draw conclusions from the following hypotheses:

(1) a intersects b in K and c in L, b and c lie in P.
(2) $a \parallel b$, a lies in P, b lies in Q, P intersects Q in d.
(3) $P \parallel Q$, R cuts P in a, and Q in b.
(4) $a \parallel b$, $b \parallel c$.
(5) $a \parallel b$, $b \parallel P$.
(6) $a \parallel P$, K lies in P, K lies on b, $b \parallel a$.
(7) $a \parallel P$, a lies in Q, Q cuts P in b.
(8) a is skew to b, $c \parallel a$, $d \parallel b$, c intersects d in K, c and d lie in P.
(9) $a \parallel P$ and Q, P intersects Q in b.
(10) a lies in both P and Q, K lies in both P and Q.
(11) $a \parallel b$, $c \parallel d$, a intersects c in K, b intersects d in L.
(12) $a \perp b$ and c at K, b and c lie in P.
(13) $a \perp P$ at K, $a \perp b$ at K.
(14) $a \perp P$, $a \parallel b$.
(15) $a \perp P$, $b \perp P$.
(16) $a \perp P$, a lies in Q.
(17) $P \perp Q$ and intersects Q in c, a lies in P, $a \perp c$.
(18) $P \perp Q$ and intersects Q in c, $a \perp Q$ at K, K lies in c.
(19) $P \perp Q$, $a \perp Q$ from K, K lies in P.
(20) $P \perp Q$, $P \perp R$, Q cuts R in a.
(21) $P \perp Q$, $a \perp Q$.

2. A road is to be built from A to B as shown on the accompanying topographic map. The curved contour lines indicate elevations at intervals of 50 ft. The scale is one-eighth inch to the mile. Draw a profile view of the road, showing the actual elevations to scale. Compute the length of the road. Where is the grade greatest?



3. On a map the horizontal distance of a mountain top A from a point B in the valley is found to be 7 miles. The elevations of A and B above sea level are respectively 5000 ft. and 2000 ft. Construct the "air line" AB and compute its length.

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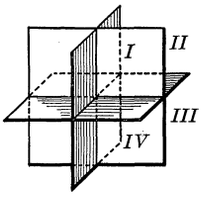
COÖRDINATES IN SPACE

582. The position of a point in space can be determined by reference to three planes, each perpendicular to the other two.

One of these planes is usually regarded as horizontal, and the other two as vertical. These three planes divide all space about their intersection point into eight parts, which are called **solid angles,** or simply **angles.** These angles are generally numbered as follows:



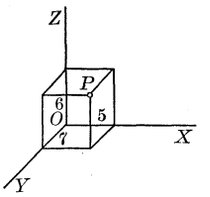
Upper, right, front,— first angle, I in figure.
Upper, right, rear,— second angle, II in figure.
Lower, right, rear,— third angle, III in figure.
Lower, right, front,— fourth angle, etc.

The first angle is generally used in geometric problems; the third angle is often used in mechanical drawing. In the following problems the first angle is used.

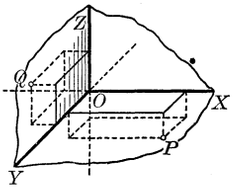
583. The point of intersection O of the three planes is called the **origin.**

The floor and two adjacent walls of a rectangular room may be used to illustrate the location of points in space.

584. Coördinates of a Point. In locating a point the planes of reference are called **coördinate planes,** and their intersections **axes of coördinates.** These axes are known as the X-, Y-, and Z-axes, as shown in the figure. (See Plane Geometry, p. 177.)



To locate a point P, whose coördinates are 5, 7, and 6, measure these distances along the X-, Y-, and Z- axes respectively, and pass through each point thus determined a plane perpendicular to the axis on which the point lies. The intersection point of these planes will be the point P.



The coördinates of a point may have negative values if distances on the X-, Y-, and Z- axes are assumed negative when laid off on the axes produced through the origin.

Thus, in the figure, the point P has the coördinates 4, 3, -1; the point Q has the coördinates -1, 4, 2. Explain how these points are located with reference to the coördinate planes.

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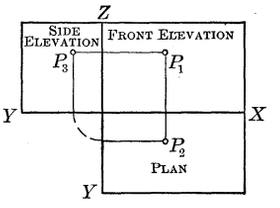
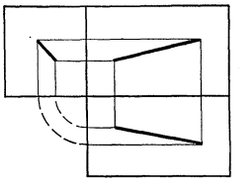
585. Plan and Elevation. Given a point in the first angle and its orthogonal projection on each of the coördinate planes. Suppose that the XY-plane and the YZ-plane are each revolved through an angle of 90° so that the three coördinate planes form one plane. The horizontal projection of a point is then called its **plan,** and the two vertical projections are called the front and side **elevations** of the point.



Thus, in the figure, P_1, P_2, P_3 are respectively the front elevation, the plan, and the side elevation of the point P (4, 2, 5).

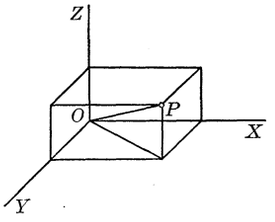


586. The plan and elevations of a line-segment are determined by the projections of its end-points. (See figure.)

EXERCISES

Describe the position of each of the points with the following coördinates:

1. (1, 2, 3).
2. (3, 5, 5).
3. (0, 4, 4).
4. (0, 0, 3).
5. (− 1, 3, 4).
6. (2, − 4, − 6).
7. (− 2, − 3, − 4).



8. Show how to apply the theorem of Pythagoras to find the distances of the points in Exs. 1–7 from the origin.

9. How are the lines joining P_1, P_2 and P_1, P_3, in the figure for § 585, situated with reference to the axes?

10. Show how, when two of the projections of a point are given, the third can be determined by construction.

11. Locate on a drawing, as in § 585, the projections of the points mentioned in Exs. 1–3.

12. Draw, as in § 586, the projections of the line-segment joining the points (1, 2, 3) and (4, 5, 6); of the line-segment joining (4, 4, 6) and (2, 2, 3). Find the length of each projection in each case.

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LOCI IN SPACE

587. The **locus in space of a point satisfying one or more given conditions** is the configuration of lines and surfaces containing all points in space which satisfy the given condition or conditions, and no other points.

Loci of points are often curved surfaces, some of which are not considered in elementary geometry. Spherical, conical, and cylindrical surfaces (§ 513), however, frequently appear as loci.

588. Loci in space are determined and established by the same methods as plane loci (§ 295).

For a locus-problem in space it is not always wise actually to construct several points on the locus. It is sufficient in most cases to imagine such points constructed, and then proceed as in plane loci, namely, form an idea of the locus, and test this conjecture (1) by showing that all points on this locus satisfy the given condition (or conditions), and (2) that all points satisfying the given condition (or conditions) lie on this locus. The order of steps (1) and (2) may be inverted.

In many cases a locus in space is obtained by the revolution of a plane locus about some fixed axis (**rotation**), in other cases by the motion of a plane locus along a fixed path, keeping it constantly parallel to its original position (**translation**).

In the following exercises formal proof need not be given.

EXERCISES

1. Sound travels 1100 feet per second in air. If a rocket explodes in the air at an elevation of 1000 feet, what is the locus of points above the earth's surface at which the explosion can be heard at the end of one-half second?

2. What is the locus of a point in a plane at a given distance from a given point? If this locus is revolved about an axis in the given plane through the given point, what surface is generated?

Theorem I. *The locus in space of a point X at a given distance d from a given point P is the spherical surface whose center is P and whose radius is d.*

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EXERCISES

1. The magnetic effect of an electric current in a wire depends upon the distance from the wire of the object affected. What is the locus of points of given field strength about a straight wire carrying a constant current?

2. A piece of work revolves on a lathe. The edge of the cutting tool moves parallel to the axis of revolution of the lathe. What is the locus of the point of contact of the tool and the work?

3. What is the locus of a point in a plane at a given distance from a given line in that plane? If this locus revolves about the line as an axis, what surface does it generate?

4. What is the locus of a point in a plane at a given distance from a given point in that plane? If the plane moves parallel to its original position so that the given point moves along a line perpendicular to the plane, what surface does the locus generate?

Theorem II. *The locus in space of a point X at a given distance d from a given line l is a cylindrical surface of revolution whose axis is l and whose radius is d.*

EXERCISES

1. What is the locus of a point 6 feet from the floor of a room? 6 feet from the ceiling?

2. A steamboat sails on an even keel over the smooth water of a bay. What is the locus of the top of its smokestack? of the bottom of its rudder-shaft?

3. What is the locus in a plane of a point at a given distance from a given line in that plane? If the given line is rotated about an axis which lies in the plane and is perpendicular to the given line, what surfaces are generated by the locus? If the given line moves along a plane perpendicular to the plane of the locus, what surfaces are generated by the locus?

Theorem III. *The locus in space of a point X at a given distance d from a given plane P consists of two planes parallel to P, one on each side of P and at the distance d from it.*

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EXERCISES

1. What is the locus of a point midway between the floor and ceiling of a room?

2. What is the locus of a point equidistant from two adjacent walls of a room?

3. A book partly open stands on a table. What is the locus of a point midway between its covers?

4. What is the locus of a point in a plane equidistant from two given parallel lines in that plane? from two given intersecting lines in the plane? If in each case by translation the given lines generate planes, what surfaces do the loci generate?

5. What is the locus of a point in a plane equidistant from two given points in that plane? If this locus revolves about an axis drawn through the two given points, what surface does it generate?

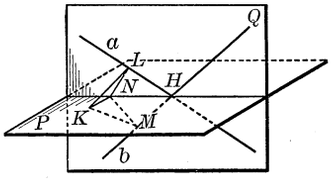
6. If the extremities of a rubber band are fastened, what is the locus of the mid-point of the band when the latter is stretched, the two halves of the band being stretched equal lengths?

Theorem IV. *The locus in space of a point equidistant from two given parallel planes is a plane parallel to the two given planes and midway between them.*

Theorem V. *The locus in space of a point equidistant from two given intersecting planes is the pair of planes bisecting the adjacent dihedral angles formed by the given planes* (§ 579).

Theorem VI. *The locus in space of a point equidistant from two given points is the plane perpendicular to and bisecting the line joining the two given points.*

Theorem VII. *The locus in space of a point equidistant from two given intersecting lines is the pair of planes each of which is perpendicular to the plane of the lines and passes through the bisector of one of the angles formed by the lines.*



To prove the last theorem, proceed as follows:

The locus in the plane of the lines is the pair of lines bisecting the angles formed by the lines. The locus in space appears to be

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the pair of planes $\perp$ to the plane of the lines and passing through the bisectors of the angles formed by the lines.

Proof. Let a and b be the given lines, intersecting at H, and P a plane $\perp$ to the plane Q of the lines a and b and cutting Q in a line bisecting the angle ab. Let K be any point in P. Draw $KL \perp$ to a and $KM \perp$ to b. Through KL pass a plane $\perp$ to a; and through KM pass a plane $\perp$ to b. These planes are each $\perp$ to Q (§ 566). Let KN be the intersection of the two planes, N lying in Q. Then $KN \perp Q$ (§ 572); also KN lies in P (§ 571). Draw LN and MN. Then $\triangle HLN \equiv \triangle HMN$ (rt. $\triangle$ h. a.); whence $\triangle KLN \equiv \triangle KMN$ (s. a. s.), and $KL = KM$. Conversely, if K is equidistant from a and b, and P is passed through K and $H \perp$ to Q, it can be shown that NH, the intersection of P and Q, bisects the angle ab. Therefore P is the required locus (§ 587).

Theorem VIII. *The locus in space of a point equidistant from two parallel lines is the plane parallel to the lines which is perpendicular to their plane and midway between them.*

(Proof similar to the foregoing.)

EXERCISES

1. A right angle revolves about one of its sides as an axis. What surface is generated by the locus of a point equidistant from the sides of the angle (§ 240)? What is the locus of a point equidistant from a plane and a line perpendicular to the plane?

2. What is the locus of a point whose distances from a plane and a line perpendicular to the plane have a fixed ratio?

3. What is the locus in space of the vertex of a right angle whose sides pass through two fixed points (§§ 286, 513)?

4. A triangle revolves about one of its sides. What is the locus of the opposite vertex?

5. What is the locus of a point at a given distance d from a given point A, and also at a given distance e from a given point B?

589. Many problems involving loci are not related directly, either to plane loci or to particular theorems of solid geometry, but require independent proof. These demonstrations are often long and involved, and may well be given informally. Many loci are determined as the intersection of two other loci.

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EXERCISES

What is the locus of a point in space

1. Equidistant from three given points?

2. Equidistant from three given parallel lines which do not lie in the same plane?

3. Equidistant from three given lines intersecting in a point, but not lying in the same plane?

4. Equidistant from three given planes which intersect in a point?

5. Equidistant from three given planes which intersect in parallel lines?

What is the locus of

6. The mid-point of a line joining any point in one of two given parallel planes to any point in the other?

7. The point which divides in a given ratio the line joining any point in one of two parallel planes to any point in the other?

What is the locus in a given plane of a point which is

8. At a given distance from a given point outside the plane?

9. At a given distance from a given parallel line outside the plane?

10. Equidistant from two given points outside the plane?

11. Equidistant from two given intersecting lines outside the plane?

What is the locus of a point in space

12. At a given distance d from one, and at a given distance e from another, of two given intersecting planes?

13. At a given distance from a given point and at a given distance e from a given plane?

14. Equidistant from two given points and at a given distance from a given plane?

15. Equidistant from two given points and at a given distance from a third point?

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BOOK VII

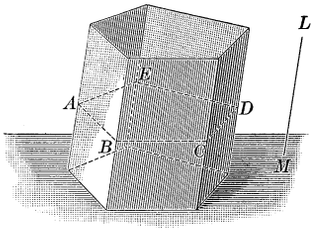
POLYHEDRONS, CYLINDERS, CONES

590. The subject matter of Book VII is the mensuration of the most common solids. Considerations of area will be followed by a study of the principal volumetric formulas. In connection with these two main topics it will be necessary to consider the plane figures which result when solids are cut by planes.

For convenience of reference, a number of definitions, previously given informally, are repeated.

591. A **polyhedron** is a solid bounded by planes.

592. The lines of intersection of the bounding planes are called the **edges;** the points of intersection of the edges, the **vertices;** and the polygons bounded by the edges, the **faces,** of the polyhedron.



593. A **section** of a solid is the plane figure formed on a plane passing through the solid.

594. A **prismatic surface** is a surface generated by a straight line, called the **generatrix,** which is supposed to move along the perimeter of a given polygon (as $ABCDE$), called the **directrix,** while constantly remaining parallel to a fixed straight line not in the plane of the polygon. When the generatrix moves completely over the perimeter of the directrix, the surface generated is a **closed prismatic surface.**

A prismatic surface may also be defined as a surface formed by a set of planes which intersect in such a way that the lines of intersection are parallel.

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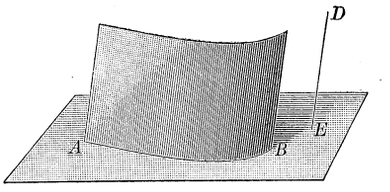
595. A **cylindrical surface** is a surface generated by a straight line, called the **generatrix,** which is supposed to move over a fixed curve, called the **directrix,** while remaining constantly parallel to a fixed straight line (not in the plane of the curve, if the curve is plane). If the directrix is a closed curve and the generatrix moves completely over it, the surface is a **closed cylindrical surface.**



596. The generatrix in any of its positions is called an **element** of the surface it generates.

597. A **prism** is a solid formed when a closed prismatic surface is cut by two parallel planes.

598. A **cylinder** is a solid formed when a closed cylindrical surface is cut by two parallel planes.

599. The figures cut on the two parallel planes in either the prism (or the cylinder) are called the **bases.**

The faces formed on the prismatic surface are called the **lateral faces,** and the intersections of the lateral faces are called the **lateral edges.** The **lateral area** of a prism is the sum of the areas of the lateral faces.

The **altitude** of a prism (or cylinder) is the perpendicular distance between the bases.

NOTE. The picture opposite represents Fingal's Cave, on the island of Staffa (Hebrides, Scotland). The cave is about 230 ft. long, 33 ft. wide, and 65 ft. high. It is one of the most perfect and beautiful specimens of natural architectural phenomena. The walls are formed by basaltic columns varying in height from 18 to 36 ft. The columns, mostly hexagonal, are of a regularity so perfect as to suggest the hand of man. A picture like this may serve as a typical instance of the fact that the fundamental solids studied in this text are merely idealized forms suggested by natural objects.

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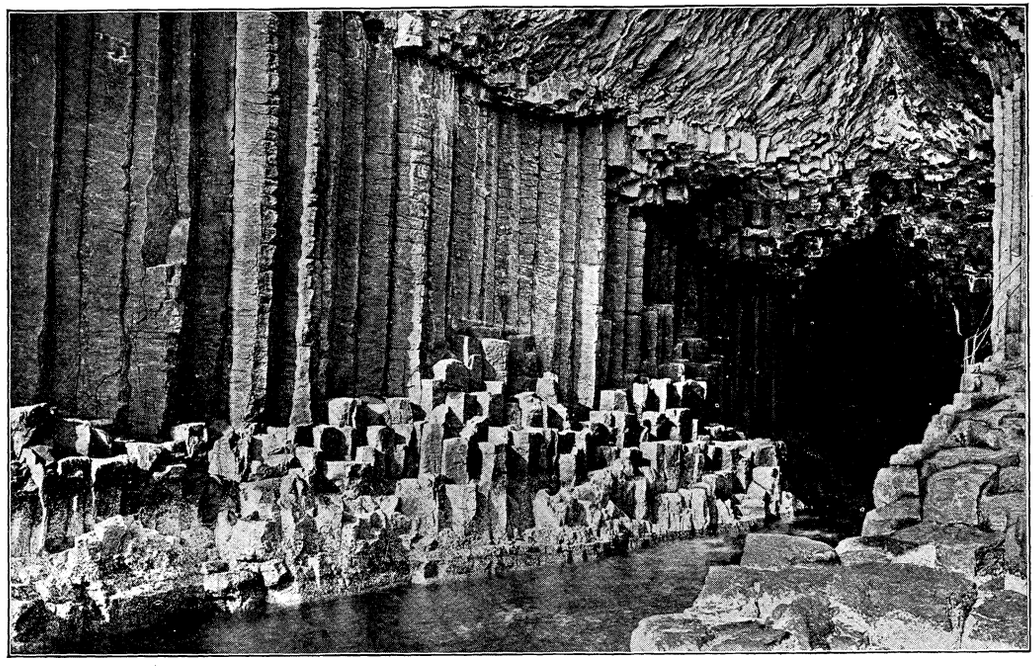

FINGAL'S CAVE

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Preliminary Propositions (I)

600. *The lateral edges of a prism (or elements of a cylinder) are equal and parallel.*

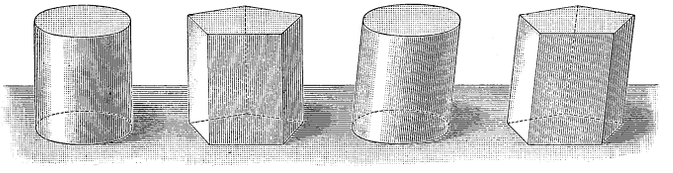
601. *The lateral faces of a prism are parallelograms.*

602. *A plane which is perpendicular to one of the lateral edges of a prism, or to one element of a cylinder, is perpendicular to all the lateral edges or the elements.*

603. A **right section** of a prism (or cylinder) is the plane figure formed when a plane cuts all the lateral edges of the prism (or elements of the cylinder) at right angles.

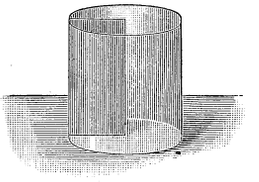
604. A **right prism** (or cylinder) is one whose lateral edges (or elements) are perpendicular to the planes of the bases.

605. An **oblique prism** (or cylinder) is one whose lateral edges (or elements) are oblique to the planes of the bases.



Right Circular Cylinder | Right Prism | Oblique Cylinder | Oblique Prism

606. Prisms Classified as to Bases. Prisms are called *triangular*, *quadrangular*, etc., according as their bases are triangles, quadrilaterals, etc.



607. A **regular prism** is a right prism whose base is a regular polygon.

608. A **circular cylinder** is a cylinder whose base is a circle. A right circular cylinder may be regarded as generated by the revolution of a rectangle about one side as an axis, and is therefore sometimes called a **cylinder of revolution.**

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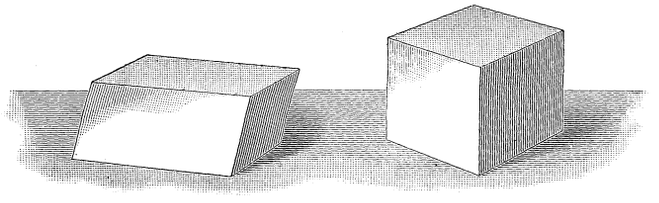
609. A **parallelepiped** is a prism whose base is a parallelogram.

610. A **right parallelepiped** is one whose lateral edges are perpendicular to the bases. A **rectangular parallelepiped** is a right parallelepiped whose base is a rectangle.

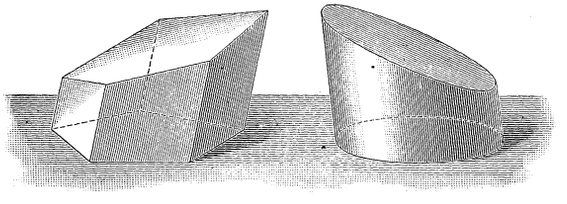
A rectangular parallelepiped is sometimes called a **cuboid**.



PARALLELEPIPED RIGHT PARALLELEPIPED

611. A **cube** is a rectangular parallelepiped, all of whose edges are equal.

612. A **truncated prism** (or truncated cylinder) is the part of a prism (or cylinder) included between the base and a section made by a plane oblique to the base.



TRUNCATED PRISM TRUNCATED CYLINDER

PRELIMINARY PROPOSITIONS (II)

613. *The section of a circular cylinder formed on the plane of two elements is a parallelogram.*

614. *Any lateral edge of a right prism (or element of a right cylinder) is equal to the altitude.*

615. *In a right prism all the lateral faces are rectangles.*

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616. *If one of the lateral edges of a parallelepiped is perpendicular to the bases, the solid is a right parallelepiped.*

617. *A parallelepiped is rectangular if the three face angles at any one of its vertices are right angles.*

618. *A parallelepiped is bounded by three pairs of parallelograms, which are situated in three pairs of parallel planes. Any one of these pairs of parallelograms may be regarded as the bases of the parallelepiped.*

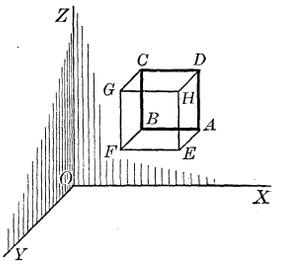
619. *The opposite faces of a cuboid are congruent rectangles.*

620. *All the faces of a cube are congruent squares.*

EXERCISES

Exercises 1–8 are introduced to give practice in sketching solids, using parallel projection (§ 545).

1. Suppose a cube to be so placed that one of its faces, $ABCD$, is parallel to the plane of projection, which may be regarded as vertical (e.g. the blackboard), and that one edge of the cube also is vertical. In the projection of this cube the face $ABCD$ appears in its original form and size. Suppose that the projecting rays are directed upon the three faces of the cube which meet at H. The edges perpendicular to the blackboard will be projected on it as equal parallel lines, AE, BF, CG, and DH, whose direction and length depend upon the direction of the rays, and may therefore be taken arbitrarily (§ 544). In the drawing take $\angle BAE = 45°$, and $AE = \frac{1}{2}AB$. These values give what is usually termed a "cabinet projection." Draw the figure.



Draw a cube.

2. Making $\angle BAE = 30°$, and using the ratio 1 : 3.

3. Making $\angle BAE = 60°$, and using the ratio 2 : 3.

4. Using cabinet projection, and supposing the diagonal AC to be vertical.

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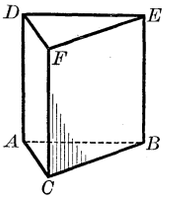
In exercises 5–8, use cabinet projection.

5. Draw a cuboid whose dimensions are 4, 5, and 6 cm.

6. Draw a regular hexagonal prism, the length of a base edge being 1 in. and the altitude being 3 in. (See Ex. 4, p. 350.)

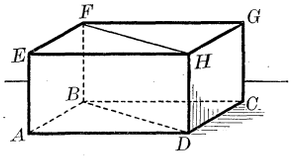
7. Draw a regular triangular prism, the base edges being 2 in. long and the altitude being 4 in. (See Ex. 2, p. 350.)



8. Draw a right circular cylinder, the diameter of the base being 2 in. and the altitude being 4 in. (See Ex. 5, p. 350.)

9. If a plane is passed through two diagonally opposite edges of a parallelepiped, prove that the resulting section is a parallelogram.

10. In the rectangular parallelepiped EC of the figure prove that the section $BFHD$ is a rectangle.



11. Prove that the diagonals of a cuboid are equal.

12. Prove that the diagonals of a parallelepiped meet in a point.

13. Make a model of a cuboid, using convenient dimensions; of a regular hexagonal prism, using the dimensions of Ex. 6; of a regular triangular prism, using the dimensions of Ex. 7.

14. On a flat sheet of paper draw a straight line. Place a model of a right prism on its side so that one of the base edges coincides with this line and one lateral face rests on the paper. Trace on the paper the perimeter of the lateral face of the prism which is in contact with the paper. Then revolve the prism about one of the lateral edges which is in contact with the paper, and cause the next base edge to coincide with the original line. Trace on the paper the next lateral face. Repeat this process until all the lateral faces have been traced on the paper. It will then be found that the resulting figure is a rectangle. (Why?) Observe that the area of this rectangle represents the lateral area of the prism.

15. Using the method of Ex. 14, draw a rectangle representing the lateral area of the prism in Ex. 6; of the prism in Ex. 7.

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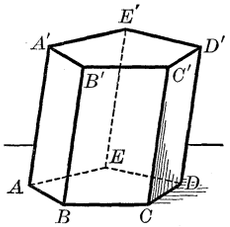
PROPOSITION I. THEOREM

621. *The bases of a prism are congruent polygons.*



Given the prism AD'.

To prove *that* $\text{base } AD \equiv \text{base } A'D'$.

Proof. 1. The quadrilateral AB' is a $\square$. Why?

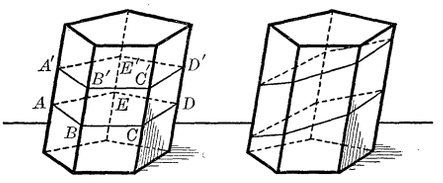
Hence $AB = A'B'$. Why?

2. Similarly, $BC = B'C'$, $CD = C'D'$, etc.

3. Also, $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc. Why?

4. Therefore $\text{base } AD \equiv \text{base } A'D'$.

622. COROLLARY 1. *Every section of a prism made by a plane parallel to the base is congruent to the base.*



623. COROLLARY 2. *All right sections of a prism are congruent polygons.*

624. COROLLARY 3. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*

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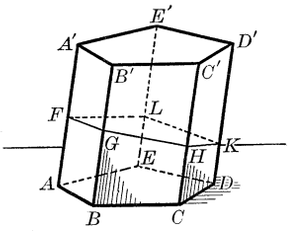
PROPOSITION II. THEOREM

625. *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*



Given the prism AD', with the right section $FGHKL$, and l the lateral area, e a lateral edge, and p the perimeter of the right section.

To prove *that* $l = ep$.

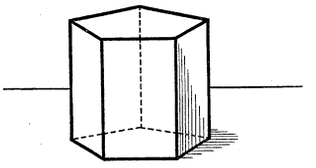
Proof. (Outline.) 1. The lateral edges may all be denoted by the same letter, e. Why?

2. The area of each lateral face equals the product of a lateral edge and a side of the right section. Why?

3. Hence $l = e(FG + GH + HK + KL + LF)$. Why?

$\therefore\ l = ep$.

626. COROLLARY. *The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.*



EXERCISES

In the following exercises the dimensions refer to right prisms, and h is the altitude, e a lateral edge, l the lateral area, and t the total area.

1. The edge of a cube is 4. Find l and t. Find the diagonal of a face, and also the interior diagonal of the cube.

2. The diagonal of a face of a cube is $\sqrt{18}$. Find e and t.

3. The diagonal of a cube is $\sqrt{21}$. Find e and t.

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4. The total area of a cube is 24. Find e.

5. If the area of a diagonal section of a cube is $\sqrt{32}$, find e and t.

6. If a, b, and c are the lengths of three concurrent edges of a rectangular parallelepiped, find t; find the length of an interior diagonal of the solid.

7. The dimensions of a rectangular room are 8, 12, and 16 ft. Find l and t.

8. A house is 30 ft. wide, 40 ft. long, 22 ft. high to the roof, 32 ft. high to the ridgepole. If the house is to be painted, how many square feet must be covered with paint if 400 sq. ft. are deducted for windows and doors?

9. The base of a prism is a right triangle whose legs are 48 and 55. The altitude is 50. Find l and t.

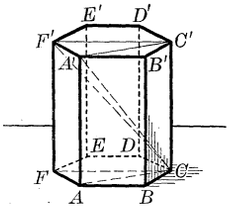
10. The lateral edge of a triangular prism is 20. The right section is an equilateral triangle of side 8. Find l.

11. A right prism has a regular hexagon for its base, and $h = 10$. Each of the base edges is 2. Find l and t.

12. The base edges of a right triangular prism are 39, 62, and 85, and $h = 60$. Find l and t.

13. A concrete octagonal pier of height 10 ft. is to be resurfaced. If each of the base edges is 1 ft., how many square feet of surfacing will be required?

14. The figure represents a regular hexagonal prism. $AB = 4$ in. and $AA' = 8$ in. Find



(1) the lengths of the lines CA and CF;

(2) the lengths of CA' and CF';

(3) the lateral area of the prism whose base is ABC and lateral edge AA';

(4) the lateral area of the prism whose base is $ABCF$ and lateral edge AA';

(5) the total area of prism ABC-A' and of prism $ABCF$-F'.

Suggestion. $\triangle FAC$ is a rt. $\triangle$. (Why?) $ACC'A'$ is a rectangle. (Why?)

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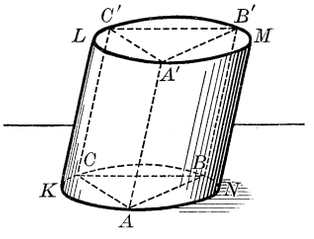
PROPOSITION III. THEOREM

627. *The bases of a cylinder are congruent.*



Given the cylinder *KM*, with bases *KN* and *LM*.

To prove *that* *base* $KN \equiv$ *base* LM.

Proof. 1. Let A and B be two fixed points in the perimeter of the lower base KN; let C be any other point in that perimeter, and let AA', BB', CC' be the corresponding elements. Draw $\triangle ABC$ and $A'B'C'$.

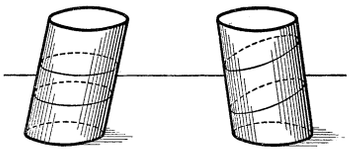
2. Then $AB = A'B'$, $BC = B'C'$, $CA = C'A'$. Why?

3. $\therefore \triangle ABC \equiv \triangle A'B'C'$. s. s. s.

4. Hence, if the base KN is placed on the base LM so that the points A and B coincide with A' and B' respectively, then C falls on C'. In like manner it can be proved that every point in one base will fall on a corresponding point in the other base.

$\therefore$ base $KN \equiv$ base LM.

628. COROLLARY. *Any section of a cylinder parallel to the base is congruent to the base, and any two parallel sections cutting all the elements are congruent.*



629. The **axis** of a circular cylinder is the line joining the centers of the bases.

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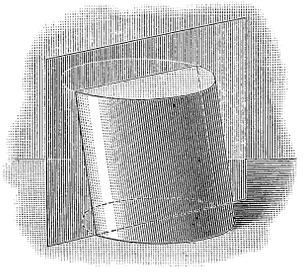
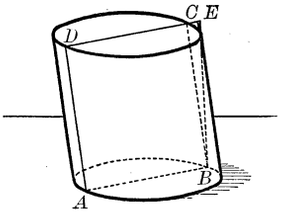
PROPOSITION IV. THEOREM

630. *The section of a circular cylinder made by a plane passing through an element is a parallelogram.*





Given *ABCD*, a section of the circular cylinder *AC* made by a plane passing through the element *AD*.

To prove *that* *ABCD is a parallelogram.*

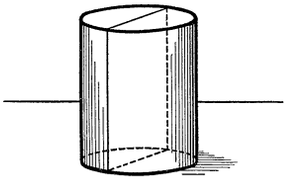
Proof. 1. Either BC is, or is not, an element. If BC is an element, the section $ABCD$ is a parallelogram (§ 613). If BC is not an element, draw the element BE.

2. Then AD and BE determine a plane section of the cylinder, and this section is a parallelogram. § 613

3. But this section must coincide with $ABCD$. § 518

4. $\therefore$ $ABCD$ is a parallelogram.

631. COROLLARY. *Every section of a right circular cylinder made by a plane passing through an element is a rectangle.*



EXERCISES

1. Show that the plane passed through any element of a circular cylinder and the center of one base will contain the axis.

2. Theorem. The center of any section of a circular cylinder parallel to the base is on the axis.

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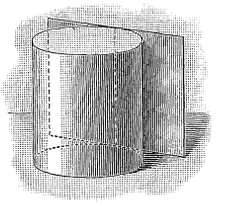
632. Tangent Plane. If a plane contains an element of a cylinder, and no other point of it, the plane is said to be **tangent** to the cylinder; the element is called the **element of contact.**

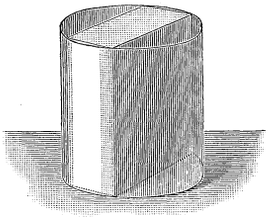
From the nature of the tangent plane it is evident that:



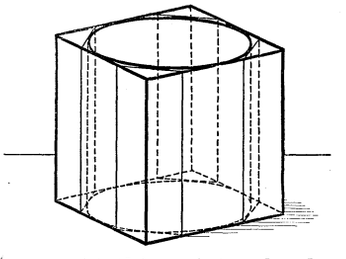
633. *A plane determined by a line that is tangent to the base of a circular cylinder and the element at the point of contact is tangent to the cylinder.*

634. *A plane which is tangent to a circular cylinder intersects the plane of each base in a straight line which is tangent to it.*

635. A prism whose lateral edges are elements of a cylinder, and whose bases are inscribed in the bases of the cylinder, is called an **inscribed prism.** The cylinder, in that case, is said to be *circumscribed about the prism.*



636. A prism whose lateral faces are tangent to the lateral surface of a cylinder and whose bases are circumscribed about the bases of the cylinder is called a **circumscribed prism.** The cylinder, in that case, is said to be *inscribed in the prism.*



637. Area of a Curved Surface. In the preceding chapters the areas of plane figures only were discussed. It is now necessary to consider areas of curved surfaces. At this point only the lateral surface of a right circular cylinder will be examined.

The unit of plane area, the square, cannot be applied to a curved surface. A method of approximation will therefore be employed, as was done in the study of the circle (Plane Geometry, pp. 309–315).

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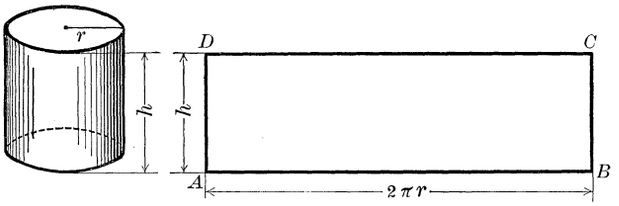
638. Lateral Area of a Right Circular Cylinder. Informal Proof. A rectangular sheet of paper, $ABCD$, may readily be rolled into the lateral surface of a right circular cylinder by superimposing the edges AD and BC so that A and D fall upon B and C respectively. Conversely, the lateral surface of a right circular cylinder may be "developed" upon a plane if the surface be cut along an element and flattened out into a rectangle.

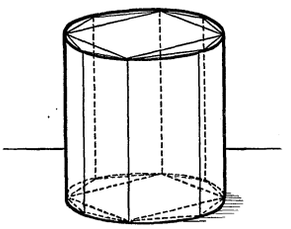


Obviously, the lateral area of the cylinder equals the area of the rectangle. The sides of the rectangle are respectively equal to the altitude and the perimeter of the base of the cylinder. It may therefore be inferred that the lateral area of a right cylinder is equal to the product of the perimeter of the base and the altitude. As applied to a right circular cylinder we have the result:

The lateral area of a right circular cylinder is equal to the product of its altitude and the circumference of the base.

In symbols, if l = lateral area, r = radius of the base, h = altitude, $\boldsymbol{l = 2\pi rh.}$

639. A second method of reaching the same conclusion is the following:



Given a right circular cylinder. Inscribe in it a regular square prism. Then inscribe right prisms whose bases are regular polygons of 8, 16, 32, $\cdots$, etc. sides. The lateral surfaces of these successive prisms evidently approach nearer and nearer to the lateral surface of the cylinder, while the

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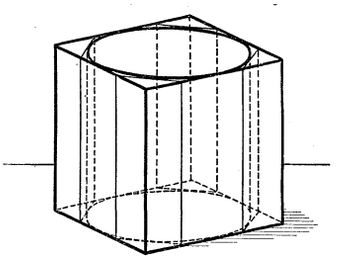
perimeters of their bases approach nearer and nearer to the circumference of the base of the cylinder (Plane Geometry, § 460). The altitude of all the prisms is the altitude of the cylinder. But the lateral area of each prism is equal to the product of its altitude and the perimeter of its base. Whence it may be inferred that the lateral area of the cylinder is equal to the product of its altitude and the circumference of its base.

The same conclusion would be reached if a series of circumscribed prisms had been constructed.

640. Lateral Area of a Right Circular Cylinder. Formal Proof. Given a right circular cylinder. Inscribe in it a right square prism. Bisect the arcs in the upper and lower bases, and complete the inscribed right prism whose base is a regular polygon of 8 sides. Continue this process (see figure, § 639), inscribing prisms whose bases are regular polygons of 16, 32, $\cdots 2^n$ sides. In like manner circumscribe a set of regular prisms of 4, 8, 16, 32, $\cdots 2^n$ sides about the cylinder.



Let l and l', respectively, denote the lateral areas of any pair of outer and inner prisms with similar bases; let p and p' denote the perimeters of the bases; and let h represent the altitude of the cylinders.

Then $l = ph$, and $l' = p'h$. Why?

Now let the number of lateral faces in each set of prisms be increased indefinitely by the process just described. Then p and p' approach a common limit c, the circumference of the base of the cylinder. (§ 487.)

Hence l and l' approach a common limit ch. (§ 489.)

These considerations justify the following

Definition. The **lateral area of a right circular cylinder** is the common limit approached by the lateral areas of inscribed and circumscribed regular prisms, as the number of lateral faces of the prisms is increased indefinitely.

641. This definition at once leads to the following

Theorem. *The lateral area of a right circular cylinder is equal to the product of its altitude and the circumference of the base.*

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EXERCISES

In the following exercises, which refer to a right circular cylinder, r denotes the radius of the base, d the diameter, h the altitude, l the lateral area, and t the total area.

1. Given $r = 5$, $h = 10$. Find l and t.

2. Given $d = 7$, $h = 20$. Find l and t.

3. Given $r = 20$, $t = 4000$. Find h and l.

4. Given $h = 10$, $l = 160$. Find r and t.

5. Given $h = 7$, $t = 924$. Find r and l.

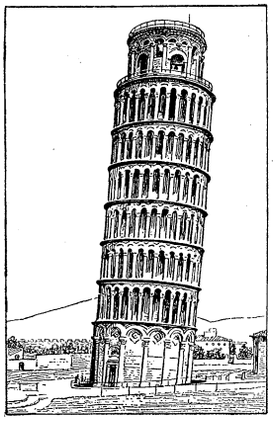
6. Given $l = 880$, $t = 2112$. Find r and h.

7. From the formula $t = 2\pi rh + 2\pi r^2$ find r in terms of h and t; find h in terms of r and t.

8. A certain factory manufactures tin cans. It received an order for 10,000 cylindrical tin cans of height 5 in. and diameter 3 in. How many square feet of tin did this order require, if 7 sq. in. are allowed for seams in each can?

9. A gas company erected a new gasholder. The gasholder was given three coats of paint. It took 50 bbl. of paint of 50 gal. each to complete the work. The height of the tank was 160 ft., and the diameter was 218 ft. If the paint cost \$1 per gallon, what was the cost per square yard of surface?

10. Two cylindrical boilers have equal altitudes, but the diameter of one is twice the diameter of the other. What is the ratio of their lateral surfaces? of the areas of their cross sections?



NOTE. The illustration shows the Leaning Tower of Pisa, at Pisa, Italy. It is a round campanile, or bell-tower, erected between 1174 and 1350. Its height is 179 ft. The six arcades above the basement have thirty columns each. The tower leans $16\frac{1}{2}$ ft. from the perpendicular. The structure is a striking example of the use of cylindrical forms in architecture.

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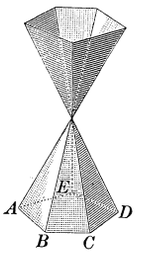
PYRAMIDS AND CONES

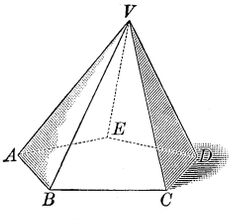
642. A **pyramidal surface** is a surface generated by a straight line, called the **generatrix,** which passes through a fixed point, and is supposed to move along the perimeter of a given polygon, called the **directrix,** which is not in the same plane as the point. When the generatrix moves completely over the perimeter, the surface generated is a **closed pyramidal surface.**



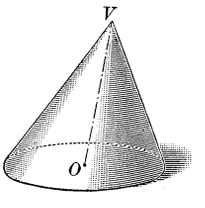
643. A **conical surface** is a surface generated by a straight line, called the **generatrix,** which passes through a fixed point, and is supposed to move along a fixed curve, called the **directrix,** which is not in the same plane as the point. When the curve is a closed curve, and the generatrix moves completely over it, the surface generated is a **closed conical surface.**

644. The fixed point is called the **vertex** of the pyramidal (or conical) surface. The moving line in any of its positions is called an **element** of the pyramidal (or conical) surface. The two parts of a pyramidal (or conical) surface on opposite sides of the fixed point are called **nappes.**

645. A **pyramid** is the solid formed when all elements of one nappe of a closed pyramidal surface are cut by a plane.



PYRAMID



CONE

646. A **cone** is the solid formed when all elements of one nappe of a closed conical surface are cut by a plane.

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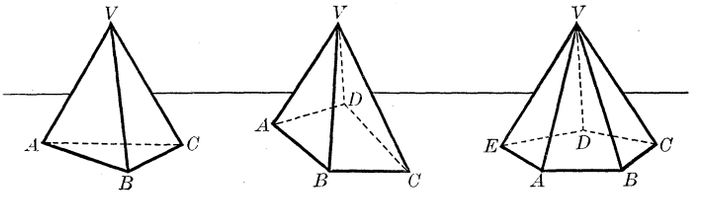
647. In either the pyramid or the cone the figure thus determined on the intersecting plane is called the **base.** The figures formed on the pyramidal surface are called **lateral faces,** and their intersections are called **lateral edges.** The **lateral area** of a pyramid is the sum of the areas of the lateral faces.

The **altitude** of a pyramid (or cone) is the perpendicular distance from the vertex to the base.

648. Pyramids classified as to Bases. Pyramids are classified as *triangular*, *quadrangular*, *pentagonal*, and so on, according as their bases are triangles, quadrilaterals, pentagons, and so on.

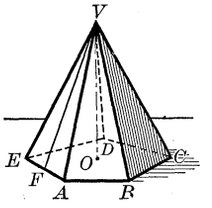


649. A cone whose base is a circle is called a **circular cone.** The straight line joining the vertex of a circular cone and the center of the base is called the **axis** of the cone.

In the figure of the circular cone in § 645, VO is the axis.

650. A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude passes through the center of the base.

651. The **slant height** of a regular pyramid is the altitude of any one of its triangular faces.



In the figure, VF is the slant height.

652. A **right circular cone** is a circular cone whose axis is perpendicular to the base; otherwise the cone is called an **oblique circular cone.**

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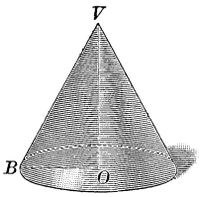
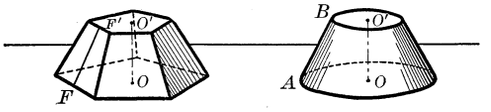
If a right triangle rotates about one of its legs as an axis, a right circular cone is generated. In the figure the hypotenuse VB generates the conical surface, and the leg OB generates the base. (Why?) A right circular cone is therefore called a **cone of revolution.**



It is evident that the hypotenuse is an element of the surface. It is sometimes called the **slant height** of the cone of revolution.

653. A **truncated pyramid** (or cone) is the portion of a pyramid (or cone) included between the base and a nonparallel plane cutting all the lateral edges of the pyramid (or elements



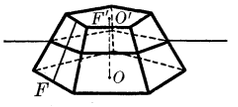
of the cone). If the cutting plane is parallel to the base, the solid is called a **frustum** of the pyramid (or cone).

The base and the parallel section are called the **bases** of the frustum. The perpendicular distance between the bases is called the **altitude** of the frustum.

The **slant height** of the frustum of a regular pyramid is the altitude of one of the trapezoidal faces of the frustum.

In the figure, OO' is the altitude and FF' is the slant height.

The **slant height** of the frustum of a right circular cone is that portion of an element of the entire cone which is intercepted between the bases of the frustum.



In the figure above, AB is the slant height of the frustum of the cone.

654. The **mid-section** of a frustum is the section determined by a plane equidistant from the bases.

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Preliminary Propositions

655. *All the lateral edges of a regular pyramid are equal.*

656. *The lateral faces of a regular pyramid are congruent isosceles triangles.*

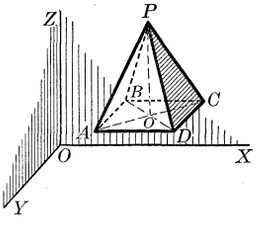
657. *The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.*

658. *The slant height of a regular pyramid (or frustum) is the same for all the lateral faces.*

659. *The section of a circular cone formed on the plane of two elements is a triangle.*

EXERCISES

1. The figure shows a parallel projection of a regular square pyramid of height 3 in. and base edge $1\frac{1}{2}$ in. Draw the projection of the base $ABCD$. Locate the center O of the base. Make the altitude OP equal to 3 in., and draw the other edges.



2. Repeat Ex. 1, making AC parallel to OX. (See Ex. 3, p. 350.)

3. Draw a regular hexagonal pyramid if the base edge is $1\frac{1}{2}$ in. and the altitude is 3 in. (See Ex. 4, p. 350.)

4. Draw a right circular cone, taking the diameter of the base 2 in. and the altitude 2 in. (See Ex. 5, p. 350.)

5. Make a model of a regular square pyramid, and develop that model on a plane. (See Ex. 14, p. 380.)

6. Draw a frustum of a regular square pyramid, if the upper and lower base edges are $1\frac{1}{2}$ in. and 2 in. and the altitude is 1 in.

7. The altitude of a regular square pyramid is h and the base edge is e. Find the lateral edge and the slant height.

8. Test Euler's theorem (Ex. 9, p. 338) in the case of a triangular prism; a hexagonal prism; a square pyramid; an octagonal pyramid; a triangular frustum. Tabulate the results obtained, and show that the formula applies in each case.

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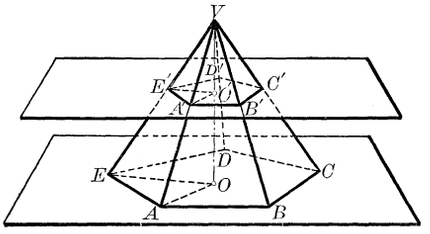
PROPOSITION V. THEOREM

660. *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and the altitude are divided proportionally.*
2. *The section made is a polygon similar to the base.*



Given the pyramid *V-ABCDE*, cut by a plane parallel to its base intersecting the lateral edges in *A'*, *B'*, *C'*, *D'*, *E'*, and the altitude *VO* in *O'*.

(a) ***To prove*** *that* $\frac{VA'}{VA} = \frac{VB'}{VB} = \frac{VC'}{VC} = \cdots \frac{VO'}{VO}$.

Proof. 1. The plane $A'D'$ is parallel to the plane AD.

2. Hence $A'B' \parallel AB$, $B'C' \parallel BC$, $\cdots$ and $A'O' \parallel AO$. Why?

3. $\therefore \frac{VA'}{VA} = \frac{VB'}{VB} = \cdots \frac{VO'}{VO}$. § 542

(b) ***To prove*** *the section $A'B'C'D'E'$ similar to the base $ABCDE$.*

Proof. (Outline.) 1. $\triangle VA'B' \sim \triangle VAB$, $\triangle VB'C' \sim \triangle VBC$, etc.

2. Hence $\frac{A'B'}{AB} = \left(\frac{VB'}{VB}\right) = \frac{B'C'}{BC} = \left(\frac{VC'}{VC}\right) = \frac{C'D'}{CD} = \cdots \frac{E'A'}{EA}$.

3. Also, the corresponding angles of the polygons are equal.

4. $\therefore A'B'C'D'E' \sim ABCDE$.

661. COROLLARY 1. *Any section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

From § 416 it follows that

$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{A'B'}^2}{\overline{AB}^2}$. But $\frac{VO'}{VO} = \frac{VA'}{VA} = \frac{A'B'}{AB}$. Hence $\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2}$.

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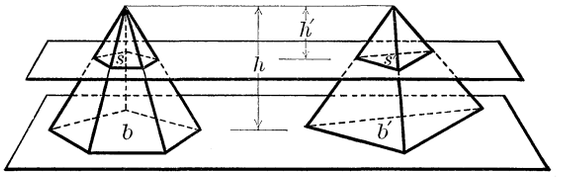
662. Corollary 2. *If two pyramids have equal altitudes and equal bases, sections made by planes parallel to the bases, at equal distances from the vertices, are equal.*

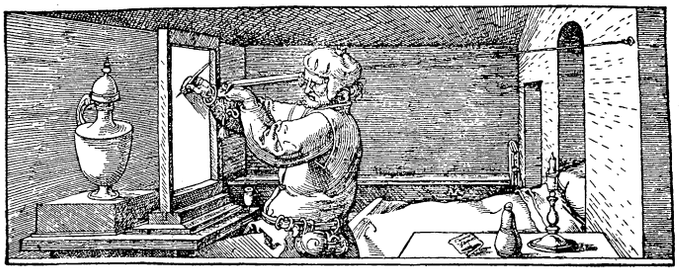


Let s and s' denote the areas of the corresponding sections; b and b', the areas of the bases; and h and h', the distances from the vertices to the sections and to the bases respectively.

Then $\frac{s}{b} = \frac{h'^2}{h^2}$, and $\frac{s'}{b'} = \frac{h'^2}{h^2}$, by Cor. 1. $\therefore \frac{s}{b} = \frac{s'}{b'}$. But $b = b'$. $\therefore s = s'$.

EXERCISES ON CENTRAL PROJECTION OR PERSPECTIVE

The methods of Parallel Projection (§ 545) are commonly used in Applied Mathematics because the actual shape and size of the object are readily suggested by the drawing. In reality, however, objects in nature appear to the eye in forms determined by Central Projection (§ 544).



Artist making a Perspective Drawing of a Vase
(From an old cut by Albrecht Dürer, 1525)

The principles underlying Central Projection can be made clear by imagining a transparent plate of glass placed between some object and the eye. Then a picture of the object on the glass plate will be

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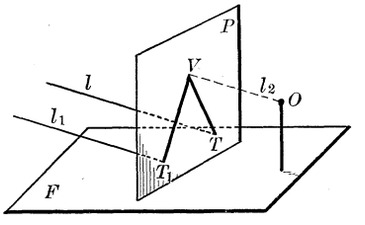
formed by the intersections with the plate of the rays joining all the points of the object to the eye of the observer. The plate (assumed to be vertical) is called the "picture plane," the horizontal plane upon which the observer stands the "ground plane," and the eye corresponds to the center of projection.

1. Vanishing Point. Let P be the picture plane, F the ground plane, and O the eye of the observer. Let the indefinite straight line l meet P at T. From O draw $l_2 \parallel$ to l, meeting P at V. Draw VT. Then VT is the projection of l on P. Explain. Again, let l_1, a line $\parallel$ to l, meet P in T_1. Then VT_1 is the projection of l_1 on P. That is, *all lines parallel to l have projections which meet at the point V, called the vanishing point for the line l or the system of lines parallel to l.*

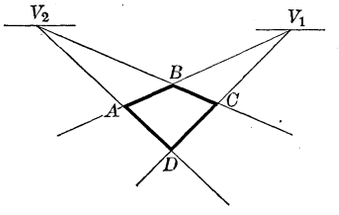


An illustration is afforded by the rails of a straight railroad track which converge as the eye follows them into the distance.

2. Parallel lines which are also parallel to the picture plane have parallel projections on that plane.

3. The plane passed through the center of projection parallel to the ground plane intersects the picture plane in a line called the "horizon line." Show that the vanishing point for any horizontal line lies in the horizon line.

4. The figure $ABCD$ is a perspective drawing of a parallelogram (V_1 and V_2 are the vanishing points of the two pairs of parallel sides of the parallelogram). Explain.



NOTE. The picture on the opposite page represents the basilica of St. Paul at Rome, Italy. It was rebuilt between 1823 and 1854 on the site of the original, ancient structure. Its imposing interior, 386 ft. long, is a scene of great splendor, due in part to the eighty huge granite columns. The picture illustrates the principles of perspective in a fine architectural setting.

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INTERIOR OF BASILICA OF ST. PAUL, ROME, ITALY

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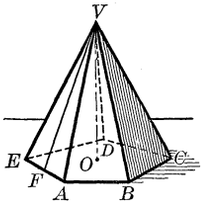
PROPOSITION VI. THEOREM

663. *The lateral area of a regular pyramid is equal to half the product of its slant height and the perimeter of its base.*



Given the regular pyramid *V-ABCDE*, with s the slant height, l the lateral area, and p the perimeter of the base.

To prove *that* $l = \frac{1}{2}sp$.

Proof. 1. The lateral faces of the pyramid are congruent triangles, with equal altitudes.

2. The area of each triangle equals $\frac{1}{2}s$ times its base. Why?

3. Hence $l = \frac{1}{2}s(AB + BC + CD \cdots)$. Why?

$$\therefore l = \tfrac{1}{2}sp.$$

664. COROLLARY 1. *The lateral area of a frustum of a regular pyramid is equal to half the product of its slant height and the sum of the perimeters of its bases.* (§ 335.)

665. COROLLARY 2. *The lateral area of a frustum of a regular pyramid is equal to the product of its slant height and the perimeter of its mid-section.* (§ 219.)

EXERCISES

1. What is true of the central projection of a polygon whose plane is parallel to the picture plane? (§ 660.)

2. The vanishing point for lines perpendicular to the picture plane is the orthogonal projection (§ 575) of the center of projection.

3. If three or more lines are parallel to a plane, the corresponding vanishing points will be in a straight line, and conversely.

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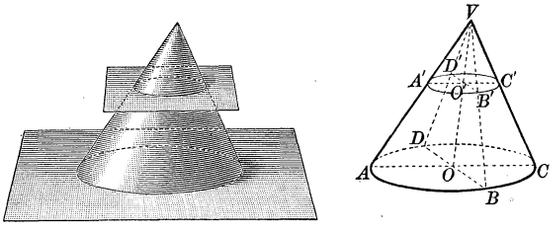
PROPOSITION VII. THEOREM

666. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Given the circular cone *V-ABCD*, with the section *A'B'C'D'* made by a plane parallel to the base.

To prove *that* $A'B'C'D'$ *is a circle.*

Proof. 1. Draw the axis VO, the radius OA, any other radius OB of the base, and the elements VA and VB. Let the planes determined by VO and VA, and by VO and VB, cut the section $A'B'C'D'$ in $O'A'$ and $O'B'$ respectively.

2. Then $O'A' \parallel OA$, and $O'B' \parallel OB$. Why?

3. $\therefore \triangle VO'A' \backsim \triangle VOA$, and $\triangle VO'B' \sim \triangle VOB$. Why?

4. $\therefore \dfrac{O'A'}{OA} = \dfrac{VO'}{VO} = \dfrac{O'B'}{OB}$. Why?

5. But $OA = OB$.

6. Therefore $O'A' = O'B'$, and the section is a circle. Why?

667. COROLLARY 1. *The axis of a circular cone passes through the center of every section which is parallel to the base.*

668. COROLLARY 2. *Any section of a circular cone parallel to the base is to the base as the square of its distance from the vertex of the cone is to the square of the altitude; and if two circular cones have equal altitudes and equal bases, sections made by planes parallel to the bases, at equal distances from the vertices, are equal.*

(Proofs identical with those of §§ 661 and 662.)

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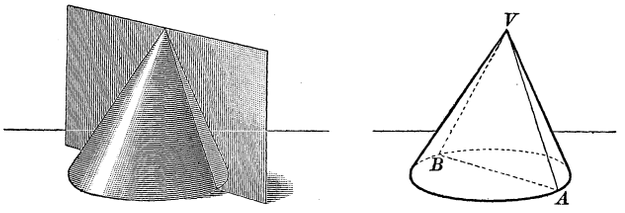
PROPOSITION VIII. THEOREM

669. *A section of a circular cone made by a plane passing through its vertex is a triangle.*



Given a circular cone, with the section *VAB* made by a plane passing through the vertex *V*.

To prove *that* *VAB is a triangle.*

Proof. 1. AB is a straight line. Why?

2. Draw the elements VA and VB.

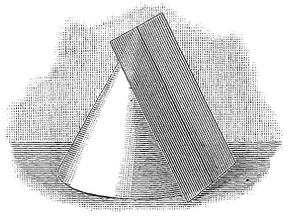
3. The plane section of the cone determined by VA and VB is a triangle. § 659

4. This section must coincide with VAB. § 518

5. Hence the section VAB is a triangle.

670. Tangent Plane. If a plane contains an element of a cone, but does not cut the surface, it is said to be **tangent** to the cone.

From the nature of a tangent plane it is evident that:



671. *A plane determined by a line which is tangent to the base of a circular cone and the element drawn to the point of contact is tangent to the cone.*

672. *If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.*

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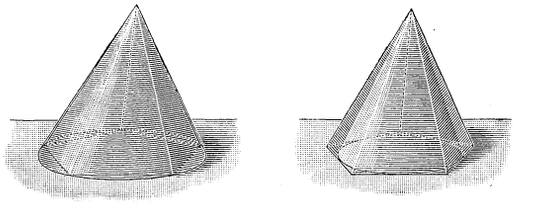
673. An **inscribed pyramid** is a pyramid whose base is inscribed in the base of a cone, and whose vertex coincides with the vertex of the cone. The cone is then said to be *circumscribed about the pyramid.*

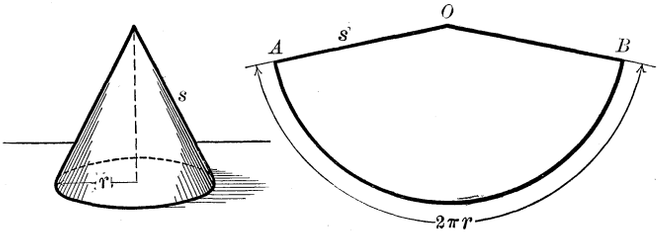


Inscribed Pyramid Circumscribed Pyramid

674. A **circumscribed pyramid** is a pyramid whose base is circumscribed about the base of a cone, and whose vertex coincides with the vertex of the cone. The cone is then said to be *inscribed in the pyramid.*

675. Measurement of the Lateral Area of a Right Circular Cone. As in the case of the cylinder, the measurement of the lateral area of a right circular cone can be obtained only by approximation.

676. Lateral Area of a Right Circular Cone. Informal Proof. If a sector of a circle OAB is cut from a sheet of paper, the



paper may be rolled into the lateral surface of a right circular cone, OA being made to coincide with OB. Conversely, the lateral surface of a right circular cone may be developed upon

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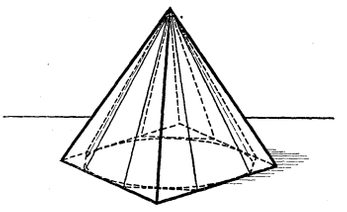
a plane, if the conical surface is cut along an element and flattened out into a sector of a circle. Obviously, the lateral area of the cone is equal to the area of the sector. The arc of the sector is equal to the circumference of the base of the cone, and its radius to an element of the cone. But the area of the sector is equal to half the product of its arc and its radius (§ 469). Hence the conclusion:

The lateral area of a right circular cone is equal to half the product of the slant height and the circumference of its base.

In symbols, if l = lateral area, r = radius of the base, and s = slant height, then $\boldsymbol{l = \pi rs}$.

677. A second method of reaching the result just stated is the following: Given a right circular cone. Circumscribe about it a regular square pyramid, then circumscribe regular pyramids whose bases are regular polygons of 8, 16, 32, $\cdots$, etc. sides. The lateral surfaces of these successive regular pyramids evidently approach nearer and nearer to the lateral surface of the given cone, while the perimeters of their bases approach nearer and nearer to the circumference of the base of the cone. The slant height of all the pyramids equals an element of the cone. But the lateral area of each pyramid equals half the product of the slant height and the perimeter of the base. Hence the above conclusion.



678. In like manner the **lateral surface of a frustum of a right circular cone** (figure, p. 402) may be developed into a sector of a circular ring, whose area is equal to the lateral area of the frustum. But the area of the sector of a circular ring may be shown to be equal to half the product of the width of

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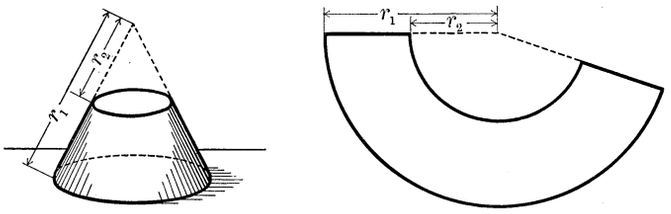
the ring (or difference of the radii of the outer and inner arcs) and the sum of the lengths of the arcs. That is,

$$A = \frac{n}{360}(\pi r_1^2 - \pi r_2^2) \qquad \S\ 468$$

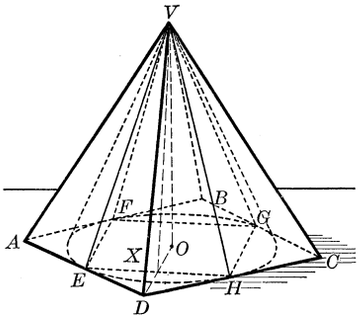
$$= \frac{n}{360}\pi(r_1 - r_2)(r_1 + r_2)$$

$$= \frac{1}{2}(r_1 - r_2)\left(\frac{n}{360}2\pi r_1 + \frac{n}{360}2\pi r_2\right). \text{ Hence}$$



The lateral area of a frustum of a right circular cone is equal to half the product of the slant height and the sum of the circumferences of the bases. § 464

679. Lateral Area of a Right Circular Cone. Formal Proof. Given a right circular cone. Inscribe in it a regular square pyramid. Bisect the arcs in the base of the cone and complete the inscribed regular pyramid whose base is a regular polygon of 8 sides. Continue this process, inscribing pyramids whose bases are regular polygons of 16, 32, $\cdots$ 2^n sides. In like manner circumscribe a set of regular pyramids of 4, 8, 16, 32, $\cdots$ 2^n sides about the cone.



Let l and l', respectively, denote the lateral areas of any pair of outer and inner pyramids with similar bases; let p and p' denote the perimeters of their bases, and s and s' their slant heights. Then

$$l = \tfrac{1}{2}ps, \text{ and } l' = \tfrac{1}{2}p's'. \qquad \text{Why?}$$

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Now let the number of lateral faces in each set of pyramids be increased indefinitely by the process just described. Then p and p' approach a common limit c, the circumference of the base of the cone (§ 487).

Also s' approaches s as a limit. For, in the figure on page 402, $VE = s$, $VX = s'$ (why?), and EH is one side of the base of the inscribed pyramid. Denote EH by a. Then $EX = \frac{1}{2}a$. Also $VX = \sqrt{\overline{VE}^2 - \overline{EX}^2}$, that is, $s' = \sqrt{s^2 - \frac{1}{4}a^2}$. But a approaches the value zero, while s remains constant, being an element of the cone. Hence $s' \doteq s$.

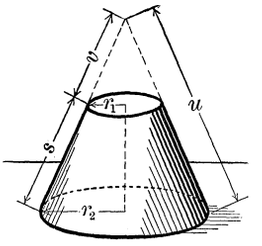
Now l approaches $\frac{1}{2}cs$ as a limit. Also l' approaches the same limit (§ 482). Hence the

Definition. The **lateral area of a right circular cone** is the common limit approached by the lateral areas of inscribed and circumscribed regular pyramids, as the number of lateral faces of the pyramids is increased indefinitely.

680. From this definition we at once obtain the

Theorem. *The lateral area of a right circular cone is equal to half the product of the slant height and the circumference of the base.*

681. Lateral Area of a Frustum of a Right Circular Cone. Formal Proof. Given the frustum F of a right circular cone. Let r_2 be the radius of the lower base, r_1 the radius of the upper base, and s the slant height. Complete the cone. Then the lateral area of F is equal to the difference of the lateral areas of two cones, whose bases have radii r_2 and r_1. Let u and v be the respective slant heights of the cones; then $u - v = s$. Also the lateral area l of the frustum is



$$l = \pi r_2 u - \pi r_1 v = \pi (r_2 u - r_1 v).$$

But $u : v = r_2 : r_1$. $\therefore\ r_1 u - r_2 v = 0$.

Whence
$$\begin{aligned} l &= \pi (r_1 u - r_2 v + r_2 u - r_1 v) \\ &= \pi (r_1 + r_2)(u - v) \\ &= \pi (r_1 + r_2)\, s. \end{aligned}$$ That is,

The lateral area of a frustum of a right circular cone is equal to half the product of the slant height and the sum of the circumferences of the upper and lower bases.

682. COROLLARY. *The lateral area of a frustum of a right circular cone is equal to the product of the slant height and the circumference of the mid-section.*

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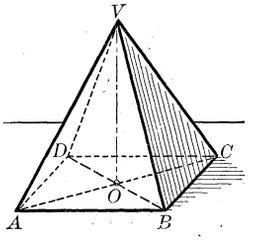
EXERCISES

LATERAL AREA. PYRAMIDS, CONES, FRUSTUMS

In the following exercises let s denote the slant height, h the altitude, l the lateral area, t the total area, e_b a base edge, e_l a lateral edge, of the solids mentioned.

1. Draw a regular square pyramid of base edge 2 in. and of height 3 in.



2. With the dimensions given in Ex. 1, find l and t.

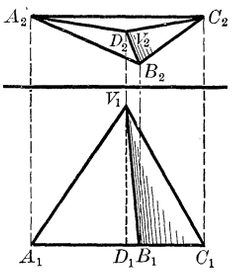
3. Given a regular hexagonal pyramid in which $e_b = 4$ and $e_l = 6$. Find l and t.

4. Given a regular hexagonal pyramid. If $e_b = 8$ cm., and $h = 6$ cm., find l and t.

5. Given a regular square pyramid. Express l in terms of s and h; in terms of e_l and e_b; in terms of h and e_b.

6. A solid consists of two congruent regular square pyramids with a common base. The distance between the opposite vertices is 16 cm., and the diagonal of the common base is 12 cm. Find t.

7. The figure represents the plan and elevation (Third Angle) of a triangular pyramid, the scale of the drawing being 1 : 5. Determine by measurement, construction, and computation the edges and the lateral area of the pyramid.



(The true length of the lateral edge VA may be found from the hypotenuse of a right triangle whose legs are V_2A_2 and V_1D_1.)

8. Find the total area of a regular tetrahedron of altitude 10. (See Ex. 36, p. 334.)

9. The Great Pyramid in Egypt originally was a regular square pyramid of height 156.4 m. and of base edge 232.2 m. At present it is really a frustum of a pyramid, its height being 147.8 m. What is the area of the platform at its top? What is the lateral area of the present structure, its irregular form being ignored?

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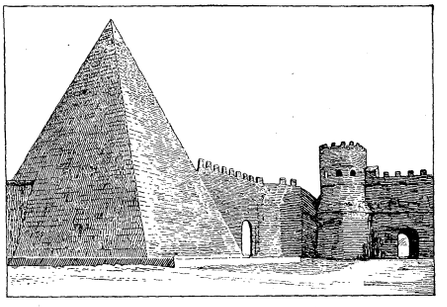
10. The monument of Cestius in Rome (shown in the illustration) is a square pyramid of height 37 m. and of base edge 30 m. How many square meters in its lateral surface?

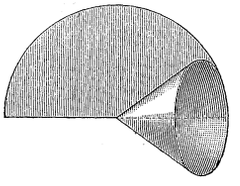


11. A rectangular building is 70 ft. long and 32 ft. wide. It has a pyramidal roof 12 ft. high, the lateral faces of the pyramid being isosceles triangles. How many tiles are needed for this roof, if 16 tiles are required to cover a square foot (no allowance for waste)?

12. Upon the faces of a cube as bases congruent regular pyramids are constructed exterior to the cube. If an edge of the cube is 4 in., and if the distance between the vertices of two opposite pyramids is 6 in., find the area of the resulting solid.

13. A right triangle of legs a and b revolves about a as an axis. If $a = 3$ in., and $b = 4$ in., find l and t of the resulting solid. Find l and t if the right triangle revolves about b as an axis; about the hypotenuse.

14. An isosceles triangle of base 6 in. and height 4 in. revolves about its base as an axis. Find the area of the resulting solid.



15. A semicircle of radius 10 cm. is rolled so as to form a cone of revolution. Find l and t of the resulting solid.

16. A quadrant of radius 14 in. is rolled so as to form a cone of revolution. Find l and t of the resulting solid.

17. A conical tent is to be 12 ft. high. The diameter of its circular base is also to be 12 ft. Making no allowance for seams, how many square yards of canvas will be used in its construction?

18. A silver cup in the form of a conical frustum is to be plated inside with gold. The upper diameter of the cup is 6 in., the diameter of the base is $4\frac{1}{2}$ in., and the height of the cup is 8 in. How many square inches of gold plating will be required for the inside of the cup?

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VOLUMES OF SOLIDS

683. The **volume** of a solid is the numerical measure of its magnitude, expressed in terms of some polyhedron as the unit.

The **unit of volume** ordinarily adopted is a cube whose edge is a standard linear unit.

684. The cubic units most commonly used are the cubic foot, the cubic inch, the cubic centimeter (cc.), and the cubic decimeter, or liter. To find the volume of a polyhedron, therefore, is to determine how many cubes of a given dimension it contains.

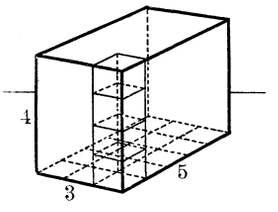
685. If two solids have the same volume, they are said to be **equal.**

686. Two polyhedrons are equal if they are congruent (that is, if they can be made to coincide) or if they can be divided into parts which are congruent in pairs.

The symbols chosen to represent the congruence or the equality of two solids are the same as the corresponding symbols in plane geometry.

687. The equality of two solids cannot always be established by dividing them into pairs of congruent parts. Other methods must frequently be used in comparing solids.

688. Volume of the Rectangular Parallelepiped. The rectangle, in plane geometry, was found to be the most convenient foundation for the subject of areas. Similarly, the rectangular parallelepiped is made the basis for the discussion of volume.



689. Consider the case in which the three dimensions* of the rectangular parallelepiped are represented by integers. In the figure let 3, 4, 5, represent the lengths of three adjoining edges of the solid in terms of the same unit. It is readily seen that the solid can be divided into as many cubic units

*The dimensions of a rectangular parallelepiped are its length, its breadth, and its height. These are the lengths of three concurrent edges.

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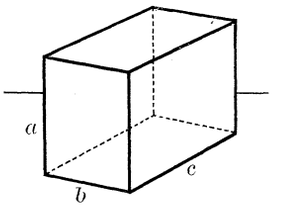
as are indicated by the product of these three integers. For the base of the solid can be divided into 3×5 squares. On each of these squares as a base a column of 4 cubes can be erected, thus giving a total of $3 \times 4 \times 5$ cubic units. This process is evidently general, so that any three integers, a, b, and c, might have been used instead of 3, 4, and 5. The volume would then be $a \times b \times c$ cubic units.



It is possible to extend these considerations to the cases in which some of the dimensions, or all of them, are represented by fractions or by irrational numbers.

This leads at once to the

Fundamental Principle. *The volume of a rectangular parallelepiped is the product of its three dimensions.*

In symbols, if v denotes the volume of the parallelepiped, and if a, b, c, are the given dimensions, then $\boldsymbol{v = abc}$.

690. Corollary 1. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*

691. Corollary 2. *Two rectangular parallelepipeds which have one dimension equal are to each other as the products of their other two dimensions.*

692. Corollary 3. *Two rectangular parallelepipeds which have two dimensions respectively equal are to each other as their third dimensions.*

693. Corollary 4. *Two rectangular parallelepipeds are equal if their three dimensions are respectively equal.*

In that case the two solids are also congruent. (Why ?)

The above corollaries may be written symbolically as follows:

$$\frac{v}{v'} = \frac{abc}{a'b'c'}; \quad \frac{v}{v'} = \frac{bc}{b'c'}[a = a']; \quad \frac{v}{v'} = \frac{c}{c'}\begin{bmatrix} a = a' \\ b = b' \end{bmatrix}; \quad v = v'\begin{bmatrix} a = a' \\ b = b' \\ c = c' \end{bmatrix}.$$

694. Corollary 5. *The volume of a rectangular parallelepiped is equal to the product of its base and altitude.*

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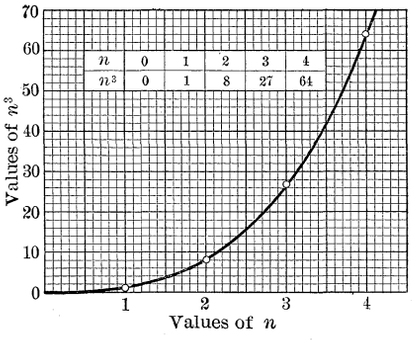
EXERCISES

1. If the volume of a cube is 8 cu. ft., its edge is evidently the **cube root** of 8. In this case the cube root is taken by inspection. When that is impossible, the cube root may be found by various methods — by arithmetic, by logarithms, or by reference to a table of cube roots (see p. 498). A satisfactory approximation can often be obtained from the *graph of cubes*. Construct this graph, using squared paper (Plane Geometry, pp. 177–179). Millimeter paper is preferable for this purpose. It is well to let one unit on the horizontal axis equal twenty units on the vertical axis.

Let n and n^3 denote any integer and its cube. Then the adjoining table represents corresponding values of these two variables, and the graph is the graph of n^3.

n	0	1	2	3	4
n^3	0	1	8	27	64



This graph may be used for both involution and evolution. Thus, if the cube of $2\frac{1}{2}$ is required, find the point representing $2\frac{1}{2}$ on the horizontal axis, and measure the vertical distance from that point to the graph. If the cube root of 10 is required, find the point representing 10 on the vertical axis and measure the horizontal distance from that point to the graph.

2. From the graph find the cubes of $2\frac{1}{2}$, $3\frac{1}{2}$, 4.1. Check by computation. Find, to one decimal place, the cube roots of 10, 15, 20, $\cdots$ to 40. Check by means of the table of cube roots.

In the following exercises e denotes a lateral edge, l the lateral area, t the total area, and v the volume.

3. If the edge of a cube is e, write the formulas for t and v. Draw graphs of t and of v, using the same axes. For what value of e is the numerical value of t equal to that of v?

4. Given a cube. Find t and v if e equals 4.1 (5.2, $3\frac{1}{5}$, 2.3). Find e if v equals 9 (6.5, 8.7, 20.32, 48). (Use the graphic method.)

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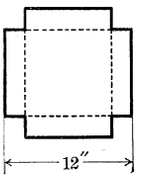
5. The edge of one cubical box is 2 (3, 4, $\cdots n$) times that of another. Compare the values of t and of v of the two boxes.

6. The diagonal section of a cube is $\sqrt{128}$. Find e, t, and v.

7. The dimensions of the floor of a rectangular room are a and b, and the height is c. Write formulas for l, t, and v.

8. The dimensions of one rectangular tank are 2 (3, 4, 5, $\cdots n$) times the dimensions of another rectangular tank. Compare the values of t and of v of the two tanks.

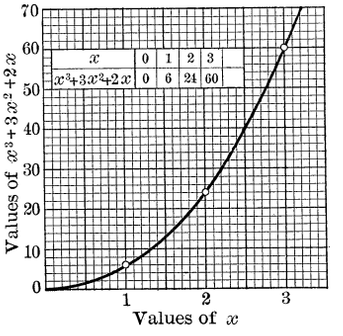
9. A square sheet of tin has four equal squares cut out at the corners, and the sides are then turned up so as to form a rectangular box. What is the volume of the box, if the side of the given sheet is 12 in., and a side of a small square is 1 in.? 2 in.? 6 in.? if the sheet of tin is rectangular, the dimensions being 2 ft. and $1\frac{1}{2}$ ft.?



10. An open tank is to be constructed of sheet iron. It is to have a square base and vertical sides. It is to contain 16 cu. ft. of water. How many square feet of material will be required for the tank if the height of the tank is 1 ft.? 4 ft.?

11. The dimensions of a rectangular box are consecutive integers. The capacity of the box is to be 60 cu. ft. Find the dimensions.

Solution. Let x, $x+1$, $x+2$ represent the dimensions. Then $x(x+1)(x+2) = 60$, from which $x^3 + 3x^2 + 2x = 60$. Draw the graph of the expression $x^3 + 3x^2 + 2x$, by putting x equal to 1, 2, 3, 4, etc. Then locate on the vertical axis the point representing 60. The horizontal distance from this point to the graph is the required value of x. Explain.



12. In Ex. 11 find the dimensions if the required volume is 24.

13. Plot a curve and solve as in Ex. 11, if the dimensions are consecutive *even* integers, and the volume is 48 (192).

14. A rectangular box is to have a square base, and its height is to be 3 ft. more than its base edge. It is to contain 54 cu. ft. Find the dimensions.

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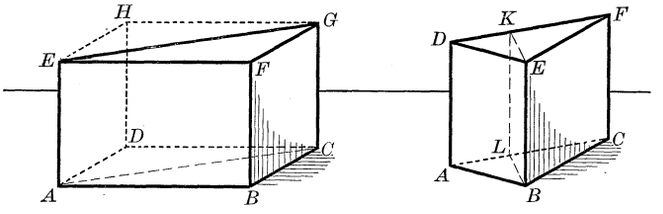
PROPOSITION IX. THEOREM

695. *The volume of any triangular right prism equals the product of its base and its altitude.*



Given the triangular right prism *ABC-E*, with base *b* and altitude *h*, and with volume *v*.

To prove *that* $v = bh$.

Proof. CASE I. *When the base is a right triangle.*

1. Complete the rectangular parallelepiped BH, by passing planes parallel to the faces AF and FC.

2. Then the prism ABC-F can be brought into coincidence with the prism ACD-H by superposition.

For the base ABC can be made to coincide with the base ACD. (Why?) Then the edge BF will coincide with DH, CG with AE, and AE with CG. Why?

3. Hence all the vertices of the given prism can be made to coincide with the corresponding vertices of the other prism.

4. But the volume of BH equals the product of its base and its altitude; therefore the volume of the triangular right prism equals the product of its base and its altitude. (Ax. 5.)

CASE II. *When the base is an oblique triangle.*

1. Suppose that AC is the longest side of the triangle ABC. Through BE pass a plane perpendicular to the face CD. Let KL be the line of intersection of the two planes.

2. Then two triangular right prisms are formed, whose bases ABL and BCL are right triangles. Why?

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3. Hence the volume of $ABL\text{-}K = \triangle ABL \cdot h$,

and the volume of $BCL\text{-}K = \triangle BCL \cdot h$.

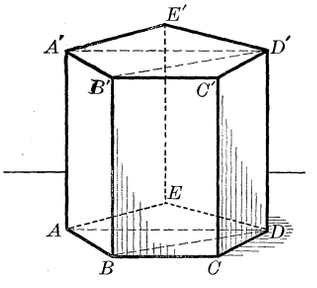
4. Therefore the volume of the given prism equals

$$\triangle ABL \cdot h + \triangle BCL \cdot h.$$

$$\therefore v = bh.$$

696. COROLLARY 1. *The volume of any right prism is equal to the product of its base and its altitude.*

By passing diagonal planes through any lateral edge, the given prism may be divided into triangular right prisms, the volume of each of which equals the product of its base and its altitude. But these prisms all have the same altitude h, and the sum of their bases equals the base of the given prism. Hence $v = bh$.



697. COROLLARY 2. *If v and v' denote the volumes of the two right prisms, b and b' their bases, and h and h' their altitudes, then*

(1) $\dfrac{v}{v'} = \dfrac{bh}{b'h'}$; (3) $\dfrac{v}{v'} = \dfrac{h}{h'}$, $[b = b']$;

(2) $\dfrac{v}{v'} = \dfrac{b}{b'}$, $[h = h']$; (4) $v = v'$, $\begin{bmatrix} h = h' \\ b = b' \end{bmatrix}$.

EXERCISES

VOLUMES OF RIGHT PRISMS

1. A hexagonal column has a height of 10 ft. and a base edge of 1 ft. Find the lateral area and the volume.

2. A swimming tank is 30 ft. long and 20 ft. wide. It is 10 ft. deep at one end and 5 ft. at the other end. How many gallons of water are required to fill it to a point which is 1 ft. from the top? (One cubic foot contains 7.4805 gal., or $7\frac{1}{2}$ gal., nearly.)

3. In digging the Panama Canal, 323,000,000 cu. yd. of material had to be removed. If all this material could have been deposited on a single pile covering an area of 100 A., how high would this pile have been? (One acre contains 43,560 sq. ft.)

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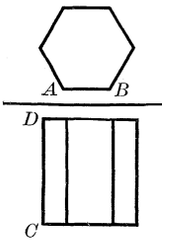
4. Each of the mammoth dredges used in the Panama Canal has a dipper the capacity of which is 15 cu. yd. If the height of this dipper is 10 ft. $9\frac{1}{2}$ in., what is the area of the bottom?

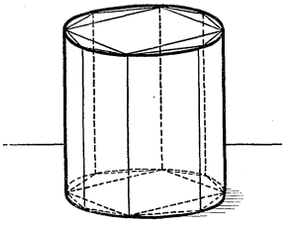
5. A cubic foot of gold was converted into 100,000,000 sq. in. of gold leaf. What was the thickness of the gold leaf?

6. The great Chinese wall is said to be 1500 mi. long, 20 ft. high, 15 ft. wide at the top, and 25 ft. wide at the bottom. If it were possible to build with this material a wall around the earth at the equator of a thickness of 1 yd., how high could it be made? (The equator is about 25,000 mi. long.)



7. The figure shows the plan and elevation (Third Angle) of a regular hexagonal prism. If AB is 2 in. and CD is 8 in., find the lateral area, the total area, and the volume.

698. Volume of a Right Circular Cylinder. Informal Proof. Inscribe in the given cylinder a prism whose base is a square and proceed as in § 639, obtaining a set of inscribed prisms whose bases are regular polygons having, respectively, 4, 8, 16, 32, ··· etc. sides. The volumes of these prisms approach nearer and nearer to the volume of the given cylinder, while the areas of their bases approach nearer and nearer to the area of the base of the cylinder. The altitude of all the prisms is the altitude of the cylinder.



The volume of each prism equals the product of the area of its base and its altitude. Hence the conclusion:

The volume of a right circular cylinder is equal to the product of its altitude and the area of its base.

In symbols, if v is the volume, r the radius of the base, and h the altitude, then

$$v = \pi r^2 h.$$

The same conclusion would be reached if a series of circumscribed prisms had been constructed.

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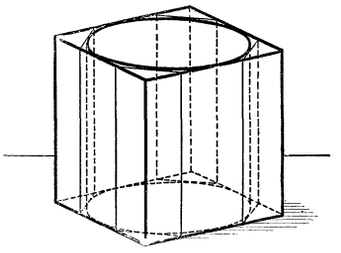
699. Volume of a Right Circular Cylinder. Formal Proof. Given a right circular cylinder. Inscribe in it right prisms whose bases are regular polygons of 4, 8, 16, $\cdots 2^n$ sides (§ 698). Also circumscribe regular prisms with bases of $4, 8, \cdots 2^n$ sides.

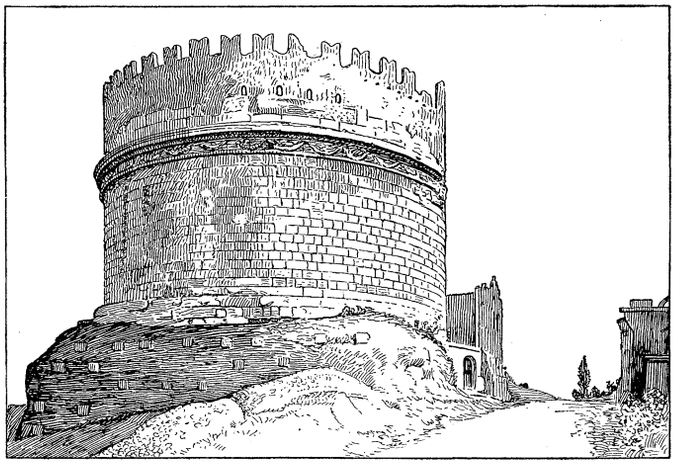


Let v and v', respectively, denote the volumes of any pair of outer and inner prisms with similar bases, and b and b' the areas of their bases. The altitude of the prisms equals the altitude of the cylinder, h. Then $v = bh$ and $v' = b'h$. (Why?)

Now let the number of lateral faces of each set of prisms be increased indefinitely (§ 640). Then b and b' approach a common limit a, the area of the base of the cylinder (§ 488). Hence v and v' approach a common limit ah (§ 489). These considerations justify the following

Definition. The **volume of a right circular cylinder** is the common limit approached by the volumes of inscribed and circumscribed regular prisms as the number of lateral faces is increased indefinitely.

700. Theorem. *The volume of a right circular cylinder equals the product of its altitude and the area of its base.*



Note. The illustration shows the tomb of Cecilia Metella on the Appian Way near Rome, having the form of a low cylindrical tower.

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EXERCISES

VOLUMES OF CYLINDERS

In the following exercises let r represent the radius of the base, d the diameter of the base, h the altitude, l the lateral area, t the total area, and v the volume of a right circular cylinder.

1. Given $r = 7$, $h = 10$. Find l, t, v.

2. Given $r = 2$ and $h = 2r$. Find l, t, v.

3. Given $t = 100$ and $h = 3r$. Find r, l, v.

4. Given $v = 1000$ and $h = 2r$. Find d.

5. From the formula $v = \pi r^2 h$ find h in terms of v and r.

6. From $v = \pi r^2 h$ find r in terms of v and h.

7. A standpipe is to contain 600,000 gal. of water. If it is to be twice as high as it is wide, what must be its dimensions?

8. Measure the dimensions of a round lead pencil, and also the thickness of the lead. What is the volume of the pencil, and what portion of this does the lead occupy?

9. Suppose the round lead pencil mentioned in Ex. 8 is to be trimmed down to a hexagonal form with the least loss of material. How long would one side of the new hexagonal base have to be? If the diameter of the pencil originally was d and its length h, work out a formula that represents the amount of loss in material in changing the form of the pencil. Apply this formula to the particular pencil you have measured. On an order of 10,000,000 pencils, how much material would be involved?

10. Assuming that a city has 400 mi. of water pipes, that the average diameter of these pipes is 1 ft., and that water is flowing through these pipes at the rate of 3 mi. per hour, how much water is required to fill this entire system, and how much water would be discharged at a particular point in the course of a day?

11. A rectangular sheet of paper whose dimensions are a feet and b feet is bent into the form of the lateral surface of a cylinder of length a feet. Find l, t, and v. If $a = 10$, and $b = 20$, what are the results? What would have been the result if b had been made the length of the cylinder?

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12. A circular storage basin is 100 ft. in diameter. During a conflagration so much water is drawn from the basin that the level of the water falls 2 ft. What is the weight of the water withdrawn? (A gallon of pure water weighs about 8.3 lb.)

13. It is claimed that through a leak not much larger than a pinhead about 1500 gal. of water will be wasted in the course of a day, under a pressure of 100 lb. At this rate what would be the loss to a city water department on such a leak in an unmetered service, if the charge for 1000 cu. ft. of water is $1?

14. The height of a hot-water boiler attached to a kitchen stove is 6 ft. The circumference of the boiler is 44 in. The thickness of the shell is $\frac{1}{16}$ in. How many gallons does the boiler contain?

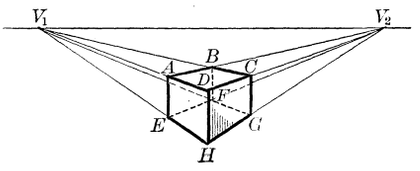
15. The **barometer** is an instrument used for the determination of atmospheric pressure. A glass tube, about a yard long, is sealed at one end and is filled with mercury. The tube is then inverted, the open end being placed in a small mercury trough. It is then found that the column of mercury sinks within the tube to a point which at sea level is very nearly 30 in. If the cross-section area of the inside of the tube were 1 sq. in., the tube would contain 30 cu. in. of mercury. This amount of mercury is held in position by the pressure of the atmosphere. If mercury weighs .49 lb. per cubic inch, how much pressure does the atmosphere exert per square inch of surface?

16. The surface of the body of a man of average proportions is about 16 sq. ft. This means that a man resists on the surface of his body an atmospheric pressure of nearly 16 T. Explain.

EXERCISES IN DRAWING

1. Make a perspective drawing of a parallelepiped.

Suggestion. Assume the lateral edges to be parallel to the picture plane. EV_2 and GV_1 intersect at F, and $BF \parallel DH$. (Why?)



2. Make a perspective drawing of a parallelepiped, no edge of which is parallel to the picture plane.

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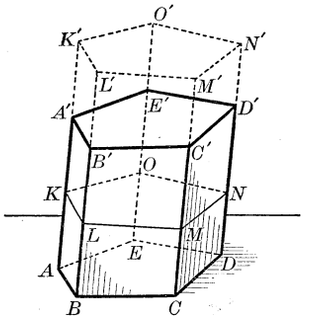
PROPOSITION X. THEOREM

701. *The volume of an oblique prism is equal to the product of its lateral edge and the area of a right section.*



Given the oblique prism AD', with the right section KN, and s the area of KN, e the length of AA', and v the volume of AD'.

To prove *that* $v = se$.

Proof. (Outline.) 1. On KN as a base construct the right prism KN', the lateral edges KK', LL', etc. being equal respectively to the lateral edges of AD'.

2. Imagine the truncated prism AN, bounded by the polygons AD and KN, to slide along the lateral edges AK' etc. until the base AD falls on the base $A'D'$. § 621

Then the plane of BK will fall on the plane of $B'K'$ (§ 565, b); AK will fall on the line $A'K'$ (§ 176, b); and K will coincide with K' (Ax. 3).

In like manner, L will coincide with L', etc.

3. Hence the solid AN will coincide with the solid $A'N'$.

4. Now if the solid $A'N'$ is taken away from the entire solid AN', the given prism AD' is left over. Similarly, by removing AN from the entire solid, the prism KN' is obtained.

5. Hence $AD' = KN'$. Why?

6. But the volume of $KN' =$ area $KN \times KK'$. Why?

7. $\therefore v = se$.

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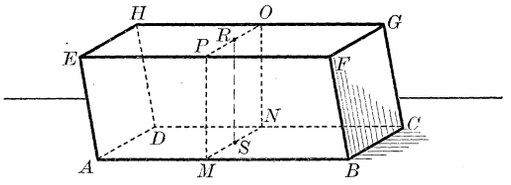
PROPOSITION XI. THEOREM

702. *The volume of an oblique parallelepiped is equal to the product of its base and its altitude.*



Given the oblique parallelepiped AG, with b the area of the base $ABCD$, h the altitude, and v the volume.

To prove *that* $v = bh$.

Proof. 1. The given parallelepiped may also be regarded as having $AEHD$ as its base, with the lateral edges AB, DC, HG, and EF. § 618

2. Draw the right section $MNOP$ by passing a plane perpendicular to AB.

3. Then $v = \text{area } MNOP \times AB$. § 701

4. But $MNOP$ is a parallelogram. Why?

Draw the altitude RS of this parallelogram. RS is also the altitude of the parallelepiped. Why?

5. Now the area of $\square MNOP = MN \times RS$. Why?

Hence $v = AB \times MN \times RS$.

6. But $AB \times MN = b$. Also $RS = h$.

$\therefore\ v = bh$.

EXERCISE

Theorem. The plane passed through two diagonally opposite edges of an oblique parallelepiped divides the parallelepiped into two equal triangular prisms.

Suggestion. Construct a right section of the parallelepiped.

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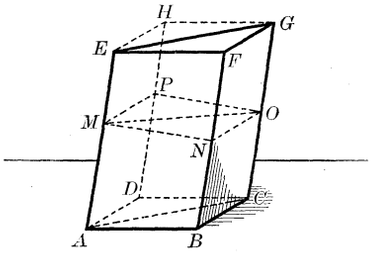
PROPOSITION XII. THEOREM

703. *The volume of an oblique triangular prism equals the product of its base and its altitude.*



Given the oblique triangular prism *ABC-EFG* with the base *b*, the altitude *h*, and the volume *v*.

To prove *that* $v = bh$.

Proof. 1. Construct the parallelogram $ABCD$, and on this parallelogram as a base construct the parallelepiped AG, the lateral edges being equal and parallel to AE. Also draw the right section $MNOP$ by passing a plane $\perp$ to AE. Let the diagonal plane CE cut this section in the line MO.

2. Then the parallelepiped AG is composed of two triangular prisms, ABC-F and ACD-H.

3. The section $MNOP$ is a $\square$, and $\triangle MNO \equiv \triangle MPO$. Why?

4. The volumes of the two triangular prisms are equal respectively to the volumes of right prisms having as their bases MNO and MPO, and altitudes equal to AE. § 701

Hence the two triangular prisms are equal, and the volume of each is half the volume of the parallelepiped.

5. But the volume of AG equals area $ABCD \times h$. § 702

Hence $v = \frac{1}{2}$ area $ABCD \times h =$ area $ABC \times h = bh$.

704. COROLLARY 1. *The volume of any prism is equal to the product of its base and its altitude.*

(Apply the method of § 696.)

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705. COROLLARY 2. *If v and v' are the volumes of any two prisms, with altitudes h and h', and bases b and b', then*

$$\frac{v}{v'} = \frac{bh}{b'h'};\ \frac{v}{v'} = \frac{h}{h'},\ [b = b'];\ \frac{v}{v'} = \frac{b}{b'},\ [h = h'];\ \text{and } v = v',\left[\begin{matrix} b = b' \\ h = h' \end{matrix}\right].$$

706. COROLLARY 3. *The volume of any circular cylinder equals the product of its base and its altitude.*

(Apply the method of §§ 698–700.)

707. The Volume of a Pyramid. If a pyramid is cut by a series of planes parallel to the base and equidistant from one another, including one through the vertex P, these parallel planes will form a series of sections similar to the base. (§ 660.)

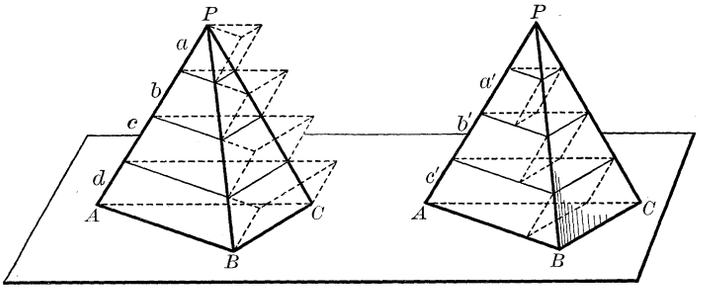

FIG. 1

FIG. 2

In the pyramid it will then be possible to construct, on the base ABC and on the sections as lower bases (Fig. 1), a series of prisms, each prism having for its altitude the distance between two consecutive parallel planes and having its lateral edges parallel to one of the edges of the pyramid. The prisms thus constructed lie partly outside of the pyramid, and are called a **set of circumscribed prisms.**

Similarly, it will be possible to construct, on the sections as upper bases (Fig. 2), prisms lying entirely within the pyramid. These inner prisms are called a **set of inscribed prisms.**

In the figures the prisms a, b, c, d, constitute one set, while the prisms a', b', c', constitute the other set.

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The meaning of the following assumption will now be clear.

ASSUMPTION. *A pyramid has a definite volume which is less than the sum of the volumes of any set of circumscribed prisms, and greater than that of any set of inscribed prisms.*

708. The circumscribed and inscribed prisms constructed in § 707 on the same section are evidently equal, for they have equal bases and equal altitudes. That is, $a = a'$, $b = b'$, and $c = c'$. But to the prism d, constructed in Fig. 1 on the base ABC, there corresponds no inscribed prism in Fig. 2.

Denote $a + b + c + d$ by s, and $a' + b' + c'$ by s'.

Then $$s - s' = d.$$

Let the number of equidistant sections be increased, let n be this number at any stage, s_n and s_n' respectively the sum of the volumes of the circumscribed and of the inscribed prisms, and l_n the volume of the circumscribed prism on the base ABC. The volume of this prism can be made small at pleasure by taking n sufficiently large.

Let	v be the volume of the pyramid.	
Then	$s_n - v < s_n - s_n'$.	§ 707
Also	$v - s_n' < s_n - s_n'$.	
But	$s_n - s_n' = l_n$.	Cons.
	$\therefore s_n - v < l_n$, and $v - s_n' < l_n$.	
Let	n increase indefinitely.	
Then	limit $s_n = v$, and limit $s_n' = v$.	§ 476

These considerations justify the following conclusion:

The volume of a pyramid is the common limit of the sums of the volumes of two sets of prisms, namely, a set of n circumscribed prisms and a corresponding set of $n - 1$ inscribed prisms, as n is increased indefinitely.

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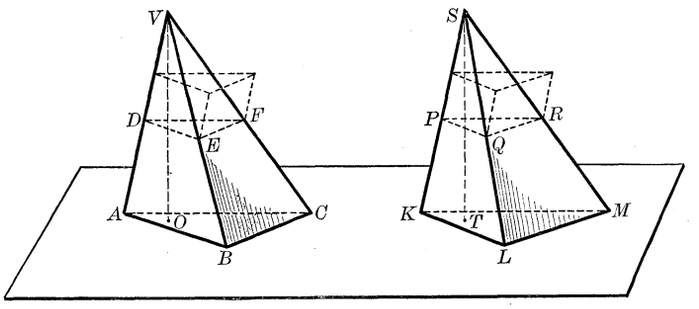
PROPOSITION XIII. THEOREM

709. *If two triangular pyramids have equal bases and equal altitudes, their volumes are equal.*



Given the pyramids *V-ABC* and *S-KLM*, having equal bases and equal altitudes.

To prove *that* *V-ABC and S-KLM have equal volumes.*

Proof. 1. Divide the altitudes VO and ST into the same number of equal parts. Using these points of division, construct a set of circumscribed prisms in connection with each pyramid. Let DEF and PQR be two corresponding sections.

2. Then section DEF equals section PQR. § 662

Therefore the prisms constructed on DEF and PQR as corresponding bases are also equal. Why?

Similarly any two corresponding prisms are equal.

3. Hence the total volumes of each of the two corresponding sets of prisms are constantly equal.

4. If the number of prisms is increased indefinitely, the volume of each pyramid is the limit of the sum of the volumes of the set of prisms associated with it. § 708

But the volumes of the two sets of prisms, being always equal variables, are really one and the same variable. Hence their limits are equal.

Therefore $V\text{-}ABC$ and $S\text{-}KLM$ have equal volumes.

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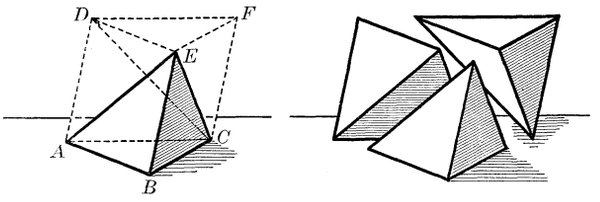
PROPOSITION XIV. THEOREM

710. *The volume of a triangular pyramid is equal to one third the product of its base and its altitude.*



Given the triangular pyramid *E-ABC*, with the volume *v*, the base *ABC* equal to *b*, and with the altitude *h*.

To prove *that* $v = \frac{1}{3}bh$.

Proof. (Outline.) 1. On the base ABC construct a triangular prism ABC-DEF, its lateral edges being equal and parallel to EB. This prism may be divided into three triangular pyramids by means of the triangular sections AEC and CED.

2. Then pyramid E-ABC = pyramid C-DEF. § 709

Likewise pyramid E-CDF = pyramid E-CAD.

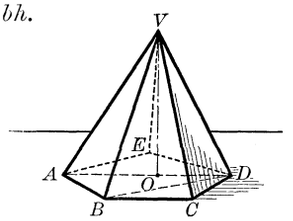
3. But the pyramid C-DEF is the same as E-CDF.

Hence E-ABC = C-DEF = E-CAD. That is, the volume of E-ABC is one third the volume of the prism.

4. But the volume of the prism = bh.

Hence the volume v of E-ABC is

$$v = \tfrac{1}{3}bh.$$



711. COROLLARY 1. *The volume of any pyramid is one third the product of its base and its altitude.*

By passing the diagonal planes VAD and VBD, the given pyramid can be divided into triangular pyramids. These triangular pyramids all have the altitude h. Also, the sum of their bases equals b. $\therefore v = \frac{1}{3}bh$.

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712. COROLLARY 2. *If v and v' are the volumes of two pyramids, with bases b and b', and altitudes h and h', then* $\frac{v}{v'} = \frac{bh}{b'h'}$; $\frac{v}{v'} = \frac{b}{b'}$, $[h = h']$; $\frac{v}{v'} = \frac{h}{h'}$, $[b = b']$; $v = v'$, $\left[\begin{matrix} b = b' \\ h = h' \end{matrix}\right]$.

EXERCISES

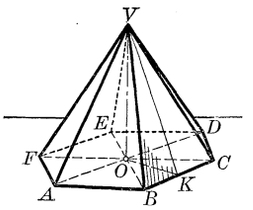
VOLUME OF PYRAMIDS

In the following exercises the abbreviations of page 404 will be used, v being the volume.

1. Given a square pyramid, $e_b = 4$, and $h = 7$. Find v.

2. Given a regular square pyramid, $s = 29$, and $h = 21$. Find v, l, and t.

3. The figure shows a regular hexagonal pyramid in parallel projection. OK is the apothem of the hexagon. VK is the slant height. Draw the figure, if $AB =$ 1 in., and $VO = 3$ in. Find v, l, and t.

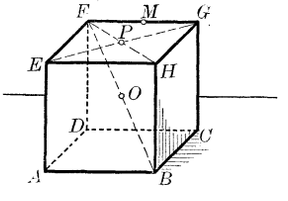


4. Given a regular hexagonal pyramid, $e_b = 6$, and $s = 6$. Find v, l, and t.

5. The edges of a triangular pyramid are all equal. If the combined length of all the edges is 18 in., find v, l, and t.

6. The three sides of the base of a triangular pyramid are 4, 13, and 15. If $h = 18$, find v.

7. The Pyramid of Cheops, in Egypt, was about 480 ft. high originally. Its base edge was about 764 ft. How many cubic yards did this square pyramid contain? If it were possible to build a wall from New York to Chicago with the stones used in the completed pyramid, how high could this wall be made, provided it had a thickness of 2 ft.?



8. The figure represents a cube. If the edge of the cube is e, what part of the volume of the cube is a pyramid whose base is $ABCD$, and whose vertex is at E? at P, the center of the upper base? at M, the mid-point of FG? at O, the mid-point of FB?

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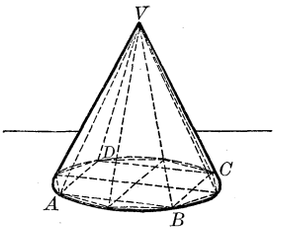
713. Volume of a Circular Cone. Informal Proof. Inscribe in the given cone a pyramid whose base is a square. Then inscribe pyramids whose bases are regular polygons having respectively 4, 8, 16, 32, $\cdots$ sides. The volumes of these pyramids will approach nearer and nearer to the volume of the given cone, while the areas of their bases approach nearer and nearer to the area of the base of the cone, the altitude of all the pyramids being the altitude of the cone. But the volume of each pyramid equals one third the product of its base and altitude. Hence the conclusion :



The volume of a circular cone is equal to one third the product of its base and altitude.

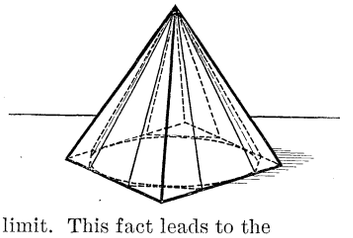
In symbols, if v is the volume, r the radius of the base, h the altitude, then

$$v = \tfrac{1}{3}\pi r^2 h.$$

The same conclusion follows if circumscribed pyramids are used.

714. Volume of a Circular Cone. Formal Proof. Given a circular cone. Inscribe pyramids whose bases are regular polygons of 4, 8, 16, $\cdots$ 2^n sides. In like manner circumscribe pyramids whose bases are regular polygons of 4, 8, 16, $\cdots$ 2^n sides. Let v and v', respectively, denote the volumes of any pair of outer and inner pyramids with similar bases. Then, with the usual notation,

$$v = \tfrac{1}{3}bh \quad \text{and} \quad v' = \tfrac{1}{3}b'h.$$



Hence it follows by the method of § 699 that v and v' have a common limit. This fact leads to the

Definition. **The volume of a circular cone** is the common limit approached by the volumes of inscribed and circumscribed pyramids, whose bases are regular polygons, as the number of lateral faces of the pyramids is increased indefinitely.

715. Theorem. *The volume of a circular cone is equal to one third the product of its base and altitude.*

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EXERCISES

VOLUME OF CONES

Each of the cones in the following problems is a right circular cone. The notation is that of page 404, v being the volume.

1. A section of a cone through the axis is an equilateral triangle of side 7 cm. Find l and v.

2. The section of a cone made by a plane passing through the axis is a triangle whose base angles are 45°. The height of the cone is 4 in. Find l, t, and v.

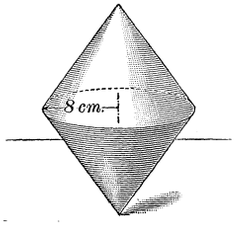
3. A right triangle of legs a and b revolves about a as an axis. Find t and v of the resulting solid.

4. What is the effect (a) on the base, (b) on the volume, of a cone, if the radius of the base is doubled while the altitude remains fixed? if the height is doubled while the radius of the base is trebled?

5. A conical vessel contains 1 l. (1 l. = 1 cu. dm. = 1000 cu. cm.) If the diameter of its base is 14 cm., find the height of the vessel.

6. A cylindrical tower has a conical top. The diameter of the tower is 6 m., the height of the vertical wall is 4 m., and the height of the tower is 7 m. What are the cubic contents of the tower?

7. The height of a solid metal cylinder is 7 cm. and the radius of its base is 8 cm. It is to be melted and cast into a double cone, the radius of the common base being 8 cm. What is the surface of the double cone?



8. A semicircle of radius $r = 8$ cm. is to be bent into a circular cone. (This means that an element of the cone equals r, and that the circumference of the base is πr.) Find the volume of the cone.

9. A quadrant of a circle of radius $r = 10$ cm. is to be bent into a circular cone. Find the volume of the cone.

10. If the conical crater of a volcano is 2 mi. in diameter, and has a depth of 1200 ft., how many cubic yards of material would be required to fill up this cavity?

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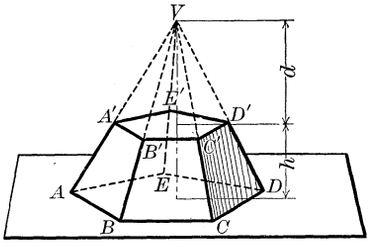
PROPOSITION XV. THEOREM

716. *The volume of a frustum of a pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Given the frustum AD', with lower base b, upper base b', and altitude h, and with volume v.

To prove *that* $v = \frac{1}{3}\,h\left(b + b' + \sqrt{bb'}\right)$.

Proof. 1. Complete the pyramid of which AD' is a frustum. Denote the volume of the entire pyramid $V\text{-}ABCDE$ by v_1, and the volume of the small pyramid $V\text{-}A'B'C'D'E'$ by v_2. Then the volume of the frustum is the difference between the volumes of the two pyramids. Let d be the altitude of the small pyramid.

2. Hence
$$\begin{aligned} v &= v_1 - v_2 \\ &= \tfrac{1}{3}\,b\,(h + d) - \tfrac{1}{3}\,b'd \\ &= \tfrac{1}{3}\,(bh + bd - b'd) \\ &= \tfrac{1}{3}\,[bh + (b - b')d]. \end{aligned}$$

3. It remains to express d in terms of b, b', and h.

But $b : b' = (h + d)^2 : d^2$. § 661

That is, $h + d : d = \sqrt{b} : \sqrt{b'}$. Why?

From which $h : d = \sqrt{b} - \sqrt{b'} : \sqrt{b'}$. Why?

Hence $$d = \frac{\sqrt{b'}}{\sqrt{b} - \sqrt{b'}} \cdot h = \frac{\sqrt{b'}\,(\sqrt{b} + \sqrt{b'})}{b - b'} \cdot h.$$

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4. Substituting this value of d in step 2,

$$v = \tfrac{1}{3}\left[bh + (b - b')\frac{\sqrt{b'}\,(\sqrt{b} + \sqrt{b'})\,h}{b - b'}\right]$$
$$= \tfrac{1}{3}\,h(b + b' + \sqrt{bb'}).$$

717. COROLLARY. *The volume of a frustum of a circular cone is equal to the sum of the volumes of three cones whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*

(Proof identical with that of § 716, with the substitution of § 668 for § 661 as an authority for step 3.)

In symbols, $v = \tfrac{1}{3}\,h\left(\pi r_1^2 + \pi r_2^2 + \sqrt{\pi^2 r_1^2 r_2^2}\right)$.

That is, $v = \tfrac{1}{3}\,\pi h\,(r_1^2 + r_2^2 + r_1 r_2)$.

EXERCISES

VOLUME OF FRUSTUMS

1. Given a frustum of a regular square pyramid of height 10 cm. The base edges are 9 cm. and 4 cm. long respectively. Find l, t, and v.

2. Solve the preceding exercise if the frustum is hexagonal.

3. Given a frustum of a right circular cone. The radii of the upper and lower bases are 6 in. and 8 in. respectively, and the height of the frustum is 12 in. Find l, t, and v.

4. The inside diameters of the bases of a flowerpot are 16 cm. and 11 cm., and the slant height is 14 cm. How many cubic centimeters of earth does it contain when it is completely filled?

5. The radii of a conical frustum are r and r'. If $r = 20$ cm., $r' = 13$ cm., and $h = 24$ cm., what are the volume and total surface of the solid resulting when a cylindrical hole of radius r' is bored through the frustum, perpendicular to the bases?

6. A cylindrical block of height 17 cm. and radius 16 cm. is to be turned down to the form of a conical frustum. One of the bases is to remain the same, but the other base is to have a radius of 10 cm. How much material has to be cut off?

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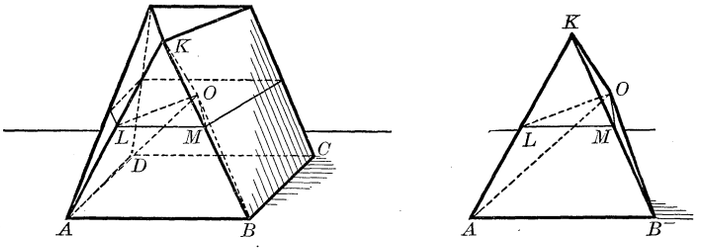
718. A **prismatoid** is a polyhedron which has for its bases two polygons in parallel planes, and for lateral faces either trapezoids, each having a side in common with each base, or triangles, each having a side in common with one base and the opposite vertex in common with the other base.

The **altitude** is the distance between the bases. The **mid-section** is the section made by a plane parallel to the bases and bisecting the altitude.

Proposition XVI. Theorem

719. *The volume of a prismatoid is equal to one sixth the product of the altitude by the sum of the bases and four times the mid-section.*



Given a prismatoid of volume v, bases b and b', mid-section m, and altitude h.

To prove *that* $v = \frac{1}{6}\, h\,(b + b' + 4\, m)$.

Proof. 1. If any lateral face is a trapezoid, divide it into two triangles by drawing a diagonal.

Let O represent any point of the mid-section. Join O to all the vertices of the solid and of the mid-section.

Then the solid is divided into a number of pyramids, all of which have O as their vertex, while their bases are b, b', and the triangles formed on the lateral faces.

2. Now the volume of the pyramid with base b is $\frac{1}{6}\, hb$, and the volume of the pyramid with base b' is $\frac{1}{6}\, hb'$. Why?

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3. It remains to consider the volumes of the lateral pyramids, such as $O\text{-}KAB$.

Since KA and KB are bisected at L and M respectively (why?), the area of $\triangle KAB$ is four times the area of $\triangle KLM$ (why?). Hence the pyramid $O\text{-}KAB$ is equal to four times the pyramid $O\text{-}KLM$. Why?

But since the pyramid $O\text{-}KLM$ is the same as the pyramid $K\text{-}LMO$, its volume is $\frac{1}{6}\ h \times \triangle LMO$.

$$\therefore\ O\text{-}KAB = \tfrac{4}{6}\ h \times \triangle LMO.$$

4. Similarly, the volume of each lateral pyramid is equal to $\frac{4}{6}\ h$ times the area of that portion of the mid-section which is included within it.

That is, the total volume of the lateral pyramids equals $\frac{4}{6}\ hm$.

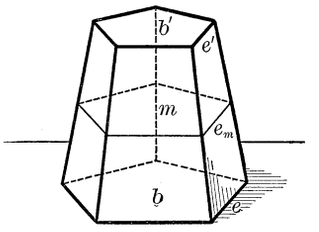
5. Therefore $v = \frac{1}{6}\ h\ (b + b' + 4\,m)$.

EXERCISES

1. From the prismatoid formula derive the formula for the volume of a prism. (Let $b = b'$.)

2. From the prismatoid formula derive the formula for the volume of a pyramid. (Let $b' = 0$.)

3. Derive from the prismatoid formula the formula for the volume of a frustum of a pyramid.



Suggestion. $b \frown m \frown b'$, by definition of frustum and § 660.

Then $\dfrac{b}{m} = \dfrac{e^2}{e_m{}^2}$ (§ 416), $\therefore\ \dfrac{\sqrt{b}}{\sqrt{m}} = \dfrac{e}{e_m}$,

and $\dfrac{b'}{m} = \dfrac{e'^2}{e_m{}^2}$, $\therefore\ \dfrac{\sqrt{b'}}{\sqrt{m}} = \dfrac{e'}{e_m}$;

whence $\dfrac{\sqrt{b} + \sqrt{b'}}{\sqrt{m}} = \dfrac{e + e'}{e_m}$; but $\dfrac{e + e'}{e_m} = 2$ (§ 219);

$$\therefore\ \frac{\sqrt{b} + \sqrt{b'}}{\sqrt{m}} = 2;\ \text{that is,}\ \sqrt{b} + \sqrt{b'} = 2\sqrt{m}.$$

Find m in terms of b and b', and substitute in the prismatoid formula.

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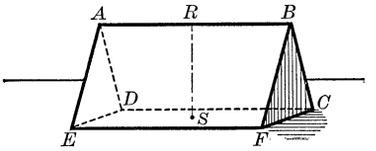
4. The frame of a wagon box has as its lower base a rectangle of dimensions 10 ft. and 6 ft., and as its upper base another rectangle whose dimensions are 12 ft. and 8 ft. The height of the frame is 8 ft. What is the capacity of the box in cubic yards?

5. The figure shows a solid called a *wedge*. It is a special case of a prismatoid. AB is parallel to the base edges CD and EF. If $CDEF$ is a rectangle of sides 8 and 10, and if AB is 7, the height RS being 9, find v.

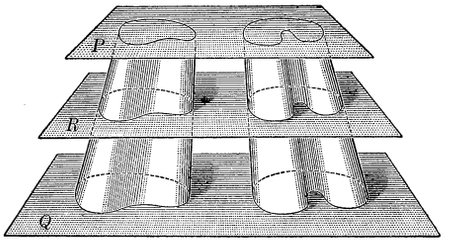


6. A water trough has the form of a wedge. The top is a rectangle of sides 12 ft. and 18 in. The depth of the trough is 16 in. How much water is there in the trough if it is filled to a depth of 10 in.?

7. Find a formula for the volume of a truncated triangular prism.

Let a, b, and c be the lateral edges and r the area of a right section of the truncated triangular prism T. The prism may be considered as a wedge by taking one of the lateral faces, k, as a base. But the values of k and m may be found in terms of a, b, c, and r, and the prismatoid formula reduces to $v = \frac{1}{3} r (a + b + c)$.

8. Cavalieri's Principle. The statement of this important principle is as follows: *If two solids are included between a pair of parallel planes, and if the two sections cut from them by any plane parallel to the including planes are equal in area, then the volumes of the solids are equal.*



The truth of the principle may be made obvious in an informal proof as follows: Divide the solids into thin slices by planes parallel to the including planes. Consider now the slices included between the same two planes. They are approximately cylindrical (or prismatic) with equal bases and altitudes, and hence approximately equal in volume. Increasing the number of slices indefinitely leads to the principle.

What theorems of Book VII may be regarded as applications of Cavalieri's Principle?

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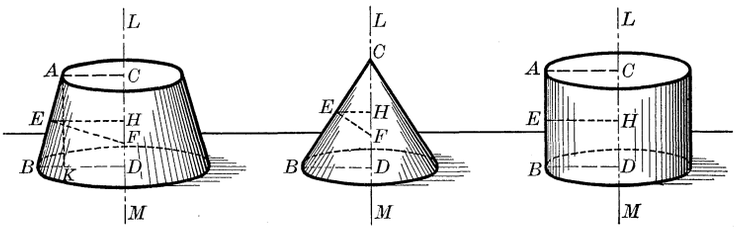
PROPOSITION XVII. THEOREM

720. *The area generated by a line-segment revolving about an axis in its plane but not crossing the axis is equal to the orthogonal projection of the line-segment upon the axis multiplied by the circumference of a circle whose radius is the perpendicular bisector of the segment terminating in the axis.*



Given the line-segment *AB* revolving about an axis *LM* in its plane, *CD* being the projection of *AB* on *LM*, *EF* being the perpendicular bisector of *AB* terminating in *LM*, and *a* being the area generated by *AB*.

To prove *that* $a = CD \times 2\pi EF$.

CASE I. *When AB is oblique to LM and does not meet LM, a is the lateral area of a frustum of a right circular cone.*

Proof. (Outline.)

1. Draw $EH \parallel$ to BD, and $AK \parallel$ to CD.
2. Then $a = AB \times 2\pi EH$. § 682
3. But $\triangle ABK \sim \triangle EFH$. Why?
4. Hence $AB : EF = AK : EH$;

that is, $AB \times EH = AK \times EF = CD \times EF$. Why?

5. Substituting, $a = CD \times 2\pi EF$.

Case II, when A lies on the axis, and Case III, when AB is parallel to the axis, to be completed.

NOTE. If $AB \perp LM$, then $EF \parallel LM$, and the proposition does not hold.

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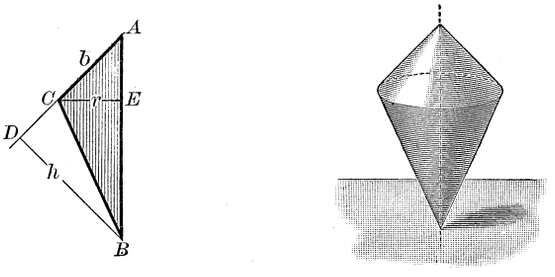
PROPOSITION XVIII. THEOREM

721. *The volume generated by a triangle revolving about one of its sides as an axis is equal to one third the product of the altitude on one of the other sides and the area generated by that side.*



Given the triangle *ABC*, revolving about the side *AB* as an axis, with the side *AC* and the altitude *BD* equal respectively to *b* and *h*, and with the volume generated equal to *v*.

To prove *that* $v = \frac{1}{3} h \times$ *the area generated by* AC.

Proof. (Outline.) 1. Draw $CE \perp$ to AB, and denote CE by r. As $\triangle ABC$ revolves, $\triangle ACE$ generates a cone, and $\triangle BCE$ also generates a cone. The two cones have a common base, and their altitudes are AE and BE respectively.

2. Hence $v = \frac{1}{3} \pi r^2 \times AE + \frac{1}{3} \pi r^2 \times BE$

$= \frac{1}{3} \pi r^2 \times AB = \frac{1}{3} \pi r \times AB \times r.$

3. But $\triangle ABD \sim \triangle ACE.$ Why?

4. Hence $AB : AC = BD : CE;$

that is, $AB \times CE = AC \times BD$, or $AB \times r = bh.$

5. Substituting, $v = \frac{1}{3} \pi r b h;$

that is, $v = \frac{1}{3} h \times \pi r b.$

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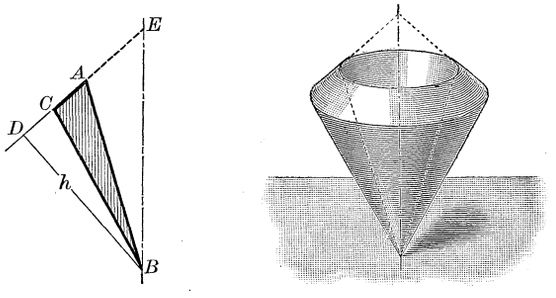
722. COROLLARY. *The volume generated by any triangle revolving about an external line in its plane and passing through a vertex is equal to one third the product of the altitude on the opposite side and the area generated by that side.*

For v = volume generated by BEC − volume generated by AEB.

Hence $v = \frac{1}{3}h \cdot$ area generated by $CE - \frac{1}{3}h \cdot$ area generated by AE
$= \frac{1}{3}h \cdot$ area generated by AC.



EXERCISES

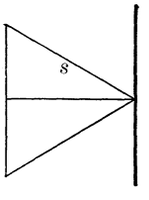
1. An equilateral triangle is revolved about its altitude h as an axis. Find l and v of the resulting solid.

2. Solve Ex. 1 if the triangle is revolved about one of its sides, s.

3. A square is revolved about one of its sides, s, as an axis. Find l and v of the resulting solid.

4. Solve the preceding problem if the square is revolved about one of its diagonals, d.

5. A regular hexagon of side s is revolved about one of its diagonals which passes through the center. Find l and v of the resulting solid.



6. If an equilateral triangle of side s revolves about an axis which passes through its vertex and is perpendicular to its altitude, as shown in the figure, find l and v of the resulting solid.

7. Repeat Ex. 6 if a square of side s revolves about an axis passing through one vertex and perpendicular to one of the diagonals.

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8. Given a regular hexagonal prism of height h and base edge e. $h = \frac{4}{3}e$. From this prism two congruent pyramids are to be removed, their bases being congruent to the bases of the prism. Find the height of either pyramid if the remaining solid is $\frac{3}{4}$ of the prism.

9. Centroids. The *center of gravity* of a material solid is commonly understood to be the point at which the weight of the solid acts. If a line, straight or curved, is regarded as a very thin homogeneous rod of uniform thickness, the center of gravity of the rod is called the **centroid** of the line. For example, the centroid of a straight line is its mid-point. Again, if a closed plane figure is regarded as a very thin homogeneous sheet (lamina) of uniform thickness, its center of gravity is called the **centroid** of the plane figure. In the same way, the **centroid** of a solid figure is defined as identical with the center of gravity of that figure, regarded as a homogeneous material solid. Thus the centroid of a spherical figure is its center.

By using the principle of the lever (Plane Geometry, Ex. 8, p. 234) it is easily shown that the center of gravity of a solid composed of two parts lies on the line-segment joining the centers of gravity of the parts and divides the segment in the inverse ratio of the weights of the parts. This result is readily extended to prove the following fundamental principle: *If a figure, plane or solid, is subdivided into parts whose centroids lie on a straight line, then the centroid of the entire figure lies on this line.*

Apply the preceding principle to a triangle by subdividing it into very narrow strips by drawing lines parallel to a side. Draw the median to this side. Then it is easy to infer that the centroid of the triangle lies on this median. Hence *the centroid of a triangle is the point of intersection of the medians.* (§§ 247–248.)

What is the centroid of a parallelogram? (§§ 201, 204.)

10. Construct the centroid of a quadrilateral by dividing it into two triangles, joining their centroids by a straight line, and dividing the line in inverse ratio to the areas of the triangles.

NOTE. The process may be applied to any rectilinear plane figure.

11. The centroid of a tetrahedron lies on a line joining any vertex to the centroid of the opposite face, and divides that line in the ratio of 3 : 1. Prove this statement.

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12. Show that the volume generated by a triangle revolving about one of its sides as an axis is equal to the area of the triangle multiplied by the length of the path of its centroid.

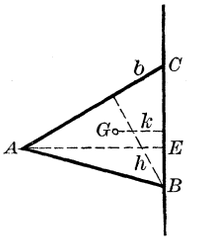
Suggestion. $v = \frac{1}{3}\,h$ *times* area generated by b

$= \frac{1}{3}\,h \times b \times \pi AE$ § 721

$= 2\pi \times AE/3 \times \frac{1}{2}\,hb.$

But $k = AE/3$ (§§ 247–248, G being the centroid).

$\therefore\ v = 2\pi k \times$ area ABC.



13. Show that the area generated by a line-segment revolving about an external axis in its plane is equal to the product of the lengths of the line and the path of its centroid.

NOTE. Exercises 12 and 13 are special cases of the **Theorems of Pappus** (300 A.D.), named from their discoverer, and also known as the **Theorems of Guldinus** (1640 A.D), who rediscovered them and made them generally known. These theorems may be stated as follows:

If a closed plane figure makes a complete revolution about an external axis in its plane,

(1) *The area of the solid generated is equal to the product of the perimeter of the plane figure and the length of the path traversed by the centroid of that perimeter;*

(2) *The volume of the solid generated is equal to the product of the area of the plane figure and the length of the path described by the centroid of that figure.*

14. Apply the Theorems of Pappus to find the area and the volume of a circular ring generated by the revolution of a circle about an external axis in its plane.

REVIEW AND MISCELLANEOUS EXERCISES

1. Prove that the sum of the squares of the four diagonals of a parallelepiped is equal to the sum of the squares of the 12 edges.

2. Prove that if a plane is passed parallel to two opposite edges of any tetrahedron, cutting the tetrahedron, the section is a parallelogram.

3. Show how a cube may be cut by a plane so that the section is an equilateral triangle; a square; a regular hexagon.

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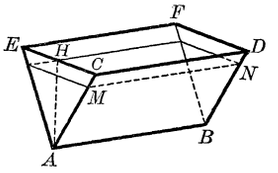
4. A plane is passed cutting three adjoining edges of a cube, so that the section is a scalene triangle. Prove that the square of the area of this triangle is equal to the sum of the squares of the areas of the three right triangles formed on the faces of the cube.

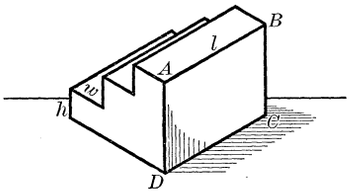
5. Prove that the volume of a tetrahedron is one sixth the product of the shortest distance between two opposite edges and the area of a parallelogram whose sides are equal and parallel to these two edges.

6. The lateral area of a regular hexagonal pyramid of edge a is n times the area of the base. Derive a formula for the height of the pyramid.

7. A water trough has the form of a triangular right prism resting on one of its lateral edges, the opposite face being horizontal. If $CD = 40$ ft., $EC = 3$ ft., AH ($\perp EC$) $= 2$ ft., find an expression for the amount of water in the trough when its depth is x. (Use the prismatoid formula.)

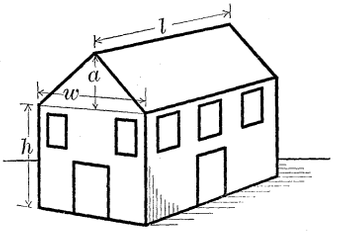


8. The figure represents a staircase of three steps. If the width of each of the treads is w, its height h, and its length l, find (a) the total surface exposed if $ABCD$ is attached to a wall, (b) the volume.



9. Solve the preceding problem if the staircase has 5 (7, 9, 11) steps.

10. In a cellar the ceiling is supported by five congruent circular columns of radius r. If the dimensions of the cellar are l, w, h (h being the height), how much of the volume of the cellar is occupied by the columns? What is the surface of these columns?



11. A rectangular house has a roof in the form of a triangular right prism. The width of the house is w, its length l, its height to the roof h, while the height of the gable triangles is a. Find a formula for the volume of the house.

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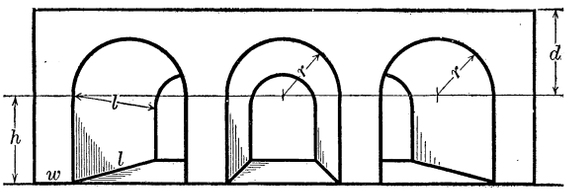
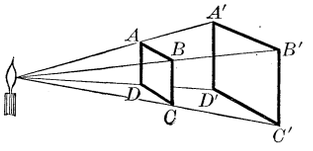
12. A stone bridge has three equal semicircular arches resting on four equal rectangular piers (figure below). If the radius of each arch is r, the dimensions of the piers l, w, h, while the distance from the top of each pier to the top of the bridge is d, write a formula for the entire volume of the bridge.

13. If in the preceding problem $l = 50$ ft., $w = 5$ ft., $h = 10$ ft., $r = 6$ ft., $d = 12$ ft., find the number of cubic yards of material used in building the bridge.



14. A principle of physics states that the illumination on a given surface caused by a fixed source of light varies inversely as the square of its distance from the light.

The diagram illustrates the principle. If the surface $A'B'C'D'$ is twice as far from the candle as the surface $ABCD$, area $A'B'C'D'$ equals four times area $ABCD$. (Why?) Hence any square unit on $A'B'C'D'$ receives one fourth the illumination received by a square unit in $ABCD$. Change the figure so that $A'B'C'D'$ is three times as far away from the candle as $ABCD$, and discuss the resulting change in intensity of illumination.



15. Two objects of the same kind are 3 ft. and 4 ft. away from a burning candle. Compare their relative brightness.

16. If the earth were three times as far away from the sun, how would the illumination at a given point compare with the present illumination?

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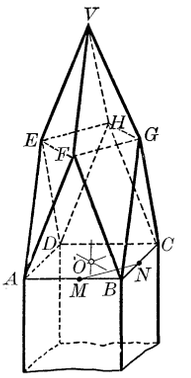
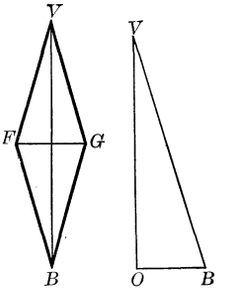
17. The figure shows a tower characteristic of certain forms of Romanesque architecture. The tower represents a square prism surmounted by a pointed top which has the following construction. The faces of the triangles ABF, BCG, CDH, DAE are continuous with the faces of the prism. The triangles are congruent. Through the adjoining sides of two consecutive triangles, such as BF and BG, planes are passed. It can be shown that these four planes must intersect in four edges concurrent at V.



Solve the following problems based on this figure:

(1) Prove that the edges formed by these planes are *concurrent*.

Suggestions. Two adjoining faces, EF and FG, intersect in an edge FV. (Why?) $AE \parallel BG$. (Why?) Hence $FV \parallel AE$ and BG. (Why?) Make $FV = AE$ and draw EV and GV. Then the $\square EF$ and FG are rhombuses. (Why?) The plane determined by E, F, G also passes through H. (Why?) Obviously, four rhombuses are thus formed, with a common vertex at V.



(2) Prove that $FG = MN$, if M and N are mid-points of two sides of the square $ABCD$.

(3) Construct the rhombus $VFBG$, if $AB = 3$ in. and $BF = 4$ in.

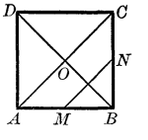
(4) Construct the altitude VO of the top of the tower.

Suggestion. In the right triangle VOB, OB is known, and VB can be constructed, being a diagonal of the rhombus $VFBG$.

(5) If $\triangle ABF$ is equilateral, and $AB = a$, find the lateral area of the part of the tower above $ABCD$.

Suggestion. $MN = \frac{1}{2}a\sqrt{2} = FG$. $VF = BF = a$.

$$\tfrac{1}{2}VB = \sqrt{a^2 - (\tfrac{1}{4}a\sqrt{2})^2} = \tfrac{1}{4}a\sqrt{14}.$$



(6) Find the formula for the volume of the tower above the section $ABCD$, if $AB = BF$, and $AB = a$.

Suggestion. Observe that V-$EFGH$ is a regular square pyramid, and that the lower part of the volume can be obtained by the *prismatoid formula.*

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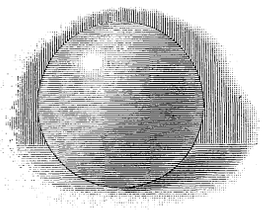
BOOK VIII

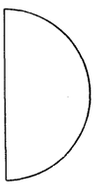
THE SPHERE

723. A **sphere** is a solid bounded by a surface all points of which are equally distant from a point within, called the **center**.

The surface of a sphere is called a **spherical surface.** The **radius** of a sphere joins any point in the surface to the center, and the **diameter** of a sphere joins two points in the surface and passes through the center. A **hemisphere** is half a sphere.



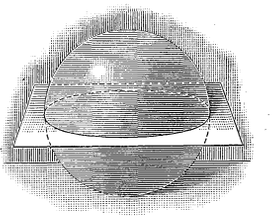
724. A spherical surface may be regarded as generated by a complete revolution of a semicircle about its diameter as an axis.



725. **Concentric spheres** have a common center. The **distance from a point to a sphere** is measured on a line passing through the center of the sphere (§§ 268–269). The **distance between two spheres** is measured on a line passing through their centers.

726. If a plane is passed through the center of a sphere, all points in the intersection of the plane and the spherical surface will be equally distant from the center of the sphere. The intersection is therefore a **circle.**

727. A **great circle** of a sphere is a circle on the surface of the sphere whose center is the center of the sphere.



728. The **axis** of a great circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle. The **poles** of a great circle are the ends of the axis. A **quadrant** is one fourth of a great circle.

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Preliminary Assumptions and Propositions

729. *A sphere may be described about any given point as a center, and with a radius equal to any given line.*

730. *Radii of the same sphere are equal.*

731. *Diameters of the same sphere are equal.*

732. *Spheres having equal radii, or equal diameters, are equal.*

733. *A point is within, on, or without a sphere, according as its distance from the center is less than, equal to, or greater than the radius.*

734. *Two spheres intersect when their line of centers is less than the sum, but greater than the difference of their radii; one sphere lies wholly within another when their line of centers is less than the difference of the radii; and two spheres lie each wholly without the other when their line of centers is greater than the sum of the radii.*

735. *Two concentric spheres are everywhere equally distant.*

736. *All great circles of a sphere are equal.*

737. *A great circle bisects the spherical surface.*

738. *Two great circles of a sphere bisect each other.*

What line have their planes in common?

739. *Through two given points on the surface of a sphere, not the ends of a diameter, one and only one great circle will pass.*

The two given points, with the center of the sphere, determine the plane of a great circle.

740. *Through the ends of a diameter of a sphere an indefinite number of great circles will pass.*

741. Assumption. *The shortest line that can be drawn on the surface of a sphere, joining two given points, is the minor arc of the great circle passed through the two given points.*

742. The **spherical distance** between two points on the surface of a sphere is the minor arc of a great circle joining the two points. This arc is sometimes called a **geodesic.**

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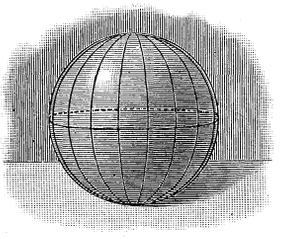
EXERCISES

1. Name the geographical great circles of the earth.

2. The length of a meridian circle of the earth is 84.2 mi. less than the length of the equator. What does this show about the shape of the earth?

(In Exs. 3 and 4 the earth is to be considered as a perfect sphere.)

3. A ship sailing from New York to Oporto, Portugal, a point almost directly east, finds its shortest route by sailing northeast past the Grand Banks. Explain this route.



4. Trace, on a geographical globe or map, the shortest line from Hammerfest, on the northern coast of Norway, to Point Barrow, Alaska.

5. If the planes of two great circles are perpendicular to each other, show that each circle passes through the poles of the other.

NOTE 1. The sphere is one of the most significant geometric forms, if only for geographic reasons. Astronomy teaches us that, like the earth, the planets and the stars are nearly spherical in shape. Moreover, the sphere is of considerable importance in all departments of applied mathematics, especially in manufacturing.

NOTE 2. The use of a few simple models will be found helpful throughout this chapter. A slated sphere, on which drawings may be made, is very desirable. For a number of propositions a globe should be at hand. A useful model of a globe may be made by means of circular rings of wire or other material. The rings should be nearly of the same diameter. One ring may be used as the equator and the others as meridians.

NOTE 3. The sphere in solid geometry corresponds to the circle in plane geometry. Many properties of the circle will suggest corresponding properties of the sphere. This fact will often aid in discovering new relations or methods of demonstration.

NOTE 4. Among the earliest scientific records are to be found observations of the apparent motions of the stars and planets, as well as of the sun and moon. These heavenly bodies were supposed to move upon the surface of a hemispherical dome, and this conception was a sufficient incentive for the early study of the properties of spherical surfaces. The earliest extant Greek mathematical document is a treatise on the sphere by an astronomer, Autolycus of Pitane, in Æolis.

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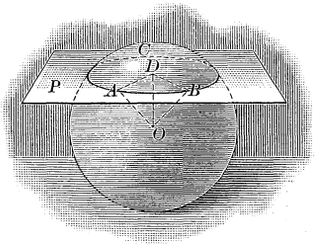
PROPOSITION I. THEOREM

743. *Every section of a spherical surface made by a plane is a circle.*



Given the sphere whose center is *O*, and *ABC*, a section of its surface made by the plane *P*.

To prove *that* ABC *is a circle.*

Proof. 1. Let A and B be any two points in the section. Draw OA and OB. Draw $OD \perp$ to P, and draw DA and DB.

2. Then $\triangle OAD \equiv \triangle OBD$. rt. $\triangle$ h. l. § 165

For OD is common,

$OD \perp DA$, and $OD \perp DB$. Why?

Also $OA = OB$. Why?

3. $\therefore DA = DB$.

4. Hence any two points A and B, and therefore all points in the section, are equally distant from D.

$\therefore ABC$ is a circle.

744. A circle on the surface of a sphere is called a **circle of the sphere.**

745. A **small circle** is a circle whose plane does not pass through the center of the sphere. The axis and poles of a small circle are defined as for a great circle (§ 728).

746. COROLLARY 1. *The axis of a small circle passes through its center; and conversely, the line which passes through the center of a small circle and the center of the sphere is the axis of the circle.*

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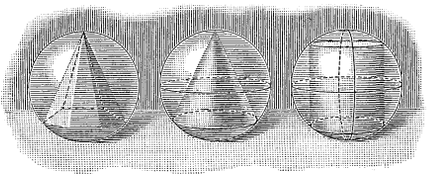
747. Corollary 2. *Circles of a sphere made by planes equally distant from the center are equal; and of two circles made by planes unequally distant from the center, the one made by the plane nearer the center has the greater radius.*

748. Corollary 3. *Circles whose planes are parallel have the same axis and the same poles.*

749. Corollary 4. *Through three given points on the surface of a sphere one and only one circle can be drawn.*

750. A polyhedron is **inscribed in a sphere** when the vertices of the polyhedron lie on the surface of the sphere. The sphere is then *circumscribed about the polyhedron.*

751. A cone is **inscribed in a sphere** when its base is a circle of the sphere and its vertex lies on the spherical surface. The sphere is then *circumscribed about the cone.*

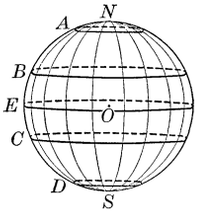


752. A cylinder (or frustum of a cone) is **inscribed in a sphere** when its bases are circles of the sphere. The sphere is then *circumscribed about the cylinder* (or *frustum*).

EXERCISES

1. Are the parallels of latitude really parallel lines? Why, then, is their name appropriate?

2. The parallels of latitude and the meridians of longitude form a network of lines encompassing the earth. What purpose do they serve?



3. Prove that the plane of a circle of a sphere is perpendicular to the planes of all great circles drawn through its poles.

4. The radius of a sphere is 20 in. Find the radius of a circle 12 in. from the center; 16 in. from the center; 18 in. from the center.

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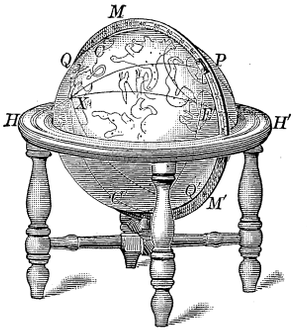
5. At what distance from the center of a sphere of radius 25 in. should a plane be passed to determine a circle of radius 7 in.? a circle of radius 15 in.? of radius 17.68 in.? of radius 20 in.? of radius 24 in.?

6. How far from the center of a sphere of radius 10 in. should a plane be passed to determine a circle whose area is one half the area of a great circle. What would be the latitude of such a circle?

7. Prove that if a right circular cone is inscribed in a sphere, the vertex is a pole of the base.

8. Find the total area of a right circular cone inscribed in a sphere of radius 10 if the slant height of the cone is equal to the diameter of its base.

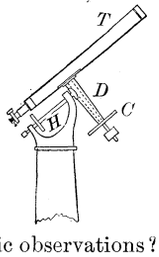
9. Show that the length of a degree of longitude is found by multiplying the length of a degree on the equator by the cosine (§ 401) of the latitude of the parallel on which the degree is measured.



Model of the Celestial Sphere

10. Review Ex. 17, p. 170, Plane Geometry, showing how the latitude of any place on the earth's surface is determined by the altitude (angle of elevation) of the polestar.

11. For the purpose of astronomical measurement the stars are regarded as lying on the surface of a sphere of very great radius, called the **celestial sphere,** with the earth as its center. The axis of the earth then points directly toward the north and south poles of this celestial sphere. The polestar is located very near the north pole. Disregarding the radius of the earth, show that the other stars, because of the daily rotation of the earth, appear to describe circles of the celestial sphere.



12. When a telescope is mounted equatorially, its principal axis points directly toward the north pole of the celestial sphere, and its secondary axis is at right angles to the principal axis. What is the advantage of equatorial mounting in making telescopic observations?

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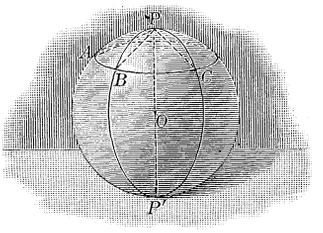
PROPOSITION II. THEOREM

753. *All points on a circle of a sphere have the same spherical distance from either of the poles of the circle.*



Given on a sphere the circle *ABC* whose poles are *P* and *P'*, and the points *A*, *B*, and *C* on this circle.

To prove *that the great circle arcs PA, PB, and PC are equal, and that the great circle arcs $P'A$, $P'B$, $P'C$ are equal.*

Proof. 1. Draw the axis PP' of $\odot ABC$.

Chords PA, PB, and PC are equal. § 554

2. $\therefore$ arcs PA, PB, and PC are equal. Why?

In like manner the great circle arcs $P'A$, $P'B$, $P'C$ are equal.

754. The spherical distance from the nearer pole to any point of a circle of a sphere is called the **polar distance** of the circle.

In the case of a great circle the distance from either pole may be taken.

755. COROLLARY. *The polar distance of a great circle is a quadrant.*

A circle may be described on the surface of a sphere by placing one point of the dividers on any point as a pole. Special "spherical compasses" with curved arms are convenient for drawing circles upon a slated sphere.

In describing a great circle the compasses span the chord of a quadrant $(=\sqrt{r^2+r^2}=r\sqrt{2})$.

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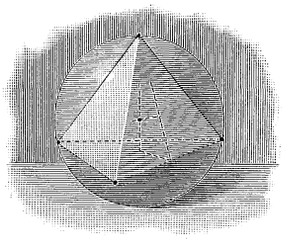
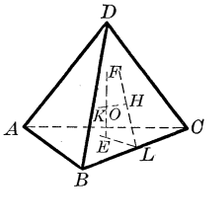
PROPOSITION III. PROBLEM

756. *To circumscribe a sphere about a given tetrahedron.*





Given the tetrahedron *ABCD*.

Required *to circumscribe a sphere about ABCD.*

Construction. 1. At E and H, the circumcenters of $\triangle ABC$ and $\triangle BCD$ respectively, erect $\perp$s EF and HK to the planes of the triangles.

2. These $\perp$s intersect at O, which is the center of the circumscribed sphere.

Proof. 1. Draw the perpendicular bisectors EL and HL of the edge BC in the planes of $\triangle ABC$ and BCD respectively.

2. Then	plane $ELH \perp BC$.	Why ?
Hence	plane $ELH \perp$ plane ABC	Why ?
and	plane $ELH \perp$ plane BCD.	Why ?
3.	$\therefore$ EF and HK lie in plane ELH.	§ 570

4. Now EF and HK are not $\parallel$, since they are respectively $\perp$ to the two intersecting lines EL and HL. § 178

$\therefore$ EF and HK meet at O.

5. Then	O is equidistant from A, B, and C.	§ 554
Also	O is equidistant from B, C, and D.	

$\therefore$ O is the center of a sphere circumscribed about $ABCD$, whose radius is OA.

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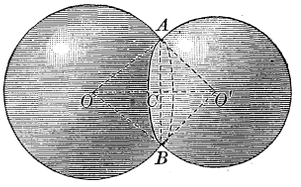
PROPOSITION IV. THEOREM

757. *If two spherical surfaces intersect, their line of intersection is a circle whose plane is perpendicular to the line of centers of the two spheres, and whose center is on that line.*



Given two intersecting spherical surfaces with centers *O* and *O'*.

To prove *that their intersection is a circle whose plane is perpendicular to OO', and whose center lies on OO'.*

Proof. 1. Let A and B be the points of intersection of two circles which, when revolved about the line of centers OO', generate the given spherical surfaces.

2. Then OO' is the $\perp$ bisector of AB at C. Why?

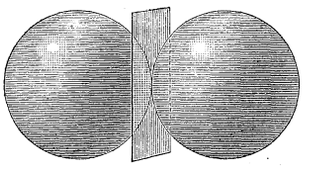
3. $\therefore$ AB generates a plane $\perp$ to OO'. § 549

4. And A describes a circle whose center is C, which is the intersection of the two spheres.

758. COROLLARY. *Through four points not in the same plane one and only one spherical surface can be passed.*

If two spherical surfaces could be passed, they would intersect in four points which (§ 757) would lie on the same circle. But this is impossible (hyp.).

759. A plane or line is **tangent to a sphere** when it has one and only one point in common with the sphere, however far the plane or line is produced.



760. Two spheres are **tangent to each other** when they are tangent to the same plane at the same point.

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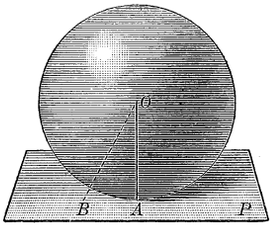
PROPOSITION V. THEOREM

761. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Given the plane *P* perpendicular to the radius *OA* of a sphere at *A*.

To prove *that P is tangent to the sphere.*

Proof. 1. Take any point B in P except A. Draw OB.

2. Then $OB > OA$. Why?

$\therefore$ B is outside the sphere.

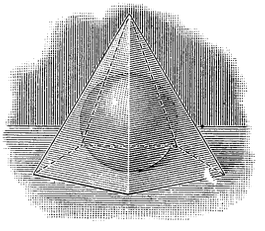
3. $\therefore$ P is tangent to the sphere. Why?

762. COROLLARY 1. *A line perpendicular to a radius at its extremity is tangent to the sphere.*

763. COROLLARY 2. *A plane or line tangent to a sphere is perpendicular to the radius at the point of contact.*

764. COROLLARY 3. *If a line and a plane are tangent to a sphere at the same point, the line lies in the plane.*

765. COROLLARY 4. *If two spheres are tangent to each other, the line of centers passes through the point of contact.*



766. A sphere is **inscribed in a polyhedron** when the faces of the polyhedron are tangent to the sphere. The polyhedron is then said to be *circumscribed about the sphere.*

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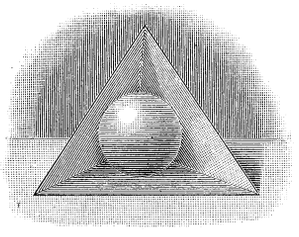
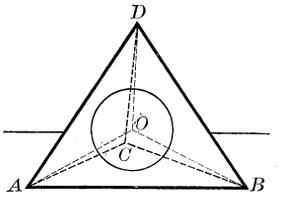
PROPOSITION VI. THEOREM

767. *To inscribe a sphere in a given tetrahedron.*





Given the tetrahedron *ABCD*.

Required *to inscribe a sphere in* $ABCD$.

Construction. 1. Bisect the dihedral angles at the edges AB, BD, and DA.

2. The point of intersection of the bisecting planes is the center of the inscribed sphere, whose radius is the perpendicular distance from this point to any face.

Proof (Informal). 1. No two of the planes can be parallel, since any two of them have a common point (A, B, or D).

2. It can be proved that any one of these planes must intersect the line of intersection of the other two, and that the planes meet in a point.

Call this point of intersection O.

3. Now O is equidistant from the planes ABC and ABD (§ 579). Also O is equidistant from the planes BDA and BDC, and from the planes DAB and DAC.

4. $\therefore$ O is equidistant from the four faces of the tetrahedron.

5. A sphere whose center is O and whose radius is the perpendicular distance from O to any face is inscribed in the tetrahedron. § 761

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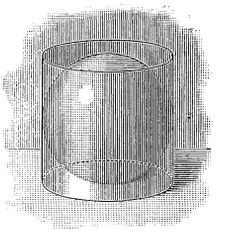
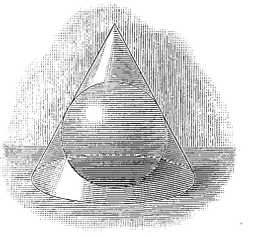


768. A sphere is **inscribed in a cylinder** when the bases of the cylinder and all elements of the cylindrical surface are tangent to the sphere. The cylinder is then said to be *circumscribed about the sphere.*



769. A sphere is **inscribed in a cone** when the base of the cone and all elements of the conical surface are tangent to the sphere. The cone is then said to be *circumscribed about the sphere.*

770. *Lines tangent to a sphere from a given external point are equal and their points of contact lie on a circle.*

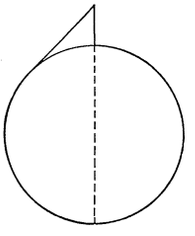
The lines may be taken (§ 275) as radii of an intersecting sphere (§ 757).

771. *The line of contact of a sphere inscribed in a cone of revolution is a small circle of the sphere.*

EXERCISES

1. Show that a sphere can be inscribed in a cube.

2. Light shines on a sphere from a point. Show that the boundary of the illuminated surface is a small circle.



3. Derive a general formula for the distance of the horizon from the top of a mountain h feet high (radius of earth = 4000 mi.).

4. The radii of two spheres are 10 and 24, and their line of centers is 26. Find the area of their circle of intersection.

5. Prove that the volume of a tetrahedron is equal to one third the product of its total area and the radius of the inscribed sphere.

6. Three balls each 6 in. in diameter lie on the floor in contact with each other, and a fourth equal ball is placed on them. Find the distance of the center of the fourth ball from the floor.

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LOCUS PROBLEMS

What is the locus of the center of a sphere

1. Passing through two given points?
2. Passing through two given points and having a given radius r?
3. Passing through three given points?
4. Tangent to two given parallel planes?
5. Tangent to two given intersecting planes?
6. Tangent to three given planes which intersect in parallel lines?
7. Tangent to three given planes which intersect in a point?
8. Tangent to three given parallel lines?
9. Tangent to three given lines which intersect in a point?
10. Tangent to the elements of a cylinder of revolution?
11. Tangent to the elements of a cone of revolution?
12. Tangent to a given plane at a given point?
13. Tangent to a given plane and having a given radius r?
14. Tangent to two concentric spheres?
15. Tangent to a given sphere of radius r and having a given radius r?

772. Area of a Sphere. The surface of the sphere, like that of the cone and the cylinder, is a curved surface, and its area cannot be measured directly in units of plane area. We can, however, approximate to the exact area, as in the next section.

Historical Note. The mensuration of the sphere is forever connected with the name of Archimedes (287–212 B.C.), of Syracuse, Sicily. He was the greatest mathematician of antiquity. His many writings include a treatise, in two books, "On the Sphere and the Cylinder." In this he proved that "the surface of a sphere is four times that of one of its great circles," and that "any sphere is four times a cone whose base is a great circle, and whose altitude is a radius of the sphere." He also found certain important relations between a sphere and the circumscribed cylinder. These discoveries became associated with his name so closely that the figure of a sphere inscribed in a cylinder is said to have been placed on his tomb. The tomb of Archimedes was still in existence at the time of Cicero, and was restored by him, in 75 B.C., during his quæstorship in Sicily.

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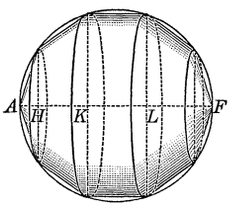
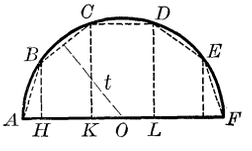
PROPOSITION VII. THEOREM

773. *The area of a sphere is equal to four times the area of a great circle.*





Given the spherical surface whose area is s, generated by the revolution of the semicircle $ABCDEF$ about the diameter AF of the sphere as an axis, and r the radius of the sphere.

To prove *that* $s = 4\pi r^2$.

Proof. (Informal.) 1. Inscribe in the semicircle half of a regular polygon of an even number of sides, as $ABCDEF$, and draw the apothems from the center O. Let t be the length of an apothem. From B, C, D, and E let fall $\perp$s to AF.

2. Then the area generated by $AB = AH \cdot 2\pi t$, § 720

the area generated by $BC = HK \cdot 2\pi t$,

etc.

3. Adding these equations,

the area generated by $ABCDEF = AF \cdot 2\pi t$. Ax. 2

$= 2r \cdot 2\pi t = 4\pi rt$.

4. Now if the number of sides of the inscribed semipolygon is indefinitely increased, it is assumed that the area generated by $ABCDEF$ approaches the area s of the sphere, and also that the apothem t approaches r.

$\therefore s = 4\pi r \cdot r = 4\pi r^2$.

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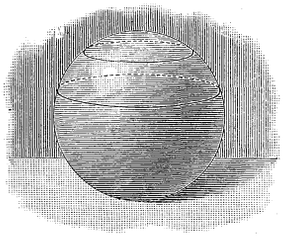
774. COROLLARY 1. *The areas of two spheres are to each other as the squares of their radii or of their diameters.*

For $$\frac{s}{s'} = \frac{4\pi r^2}{4\pi r'^2}; \quad \therefore \frac{s}{s'} = \frac{r^2}{r'^2}.$$

775. A **zone** is the portion of the surface of a sphere included between two parallel planes which meet the sphere. The circles in which the planes cut the sphere are called the **bases** of the zone. If one of the planes is tangent to the sphere, the zone is called a **zone of one base.** The **altitude** of a zone is the perpendicular distance between the planes of the bases.



Name the geographical zones. Which are zones of one base?

776. COROLLARY 2. *The area of a zone is equal to the product of the altitude of the zone and the circumference of a great circle of the sphere.*

In § 773, if the arc BCD alone is considered, it generates a zone whose area, s, is equal to $HL \cdot 2\pi r$; that is,

$$s = 2\pi rh,$$

where h is the altitude of the zone.

777. COROLLARY 3. *Zones on the same sphere or on equal spheres are to each other as their altitudes.*

778. COROLLARY 4. *The area of a zone of one base is equal to the area of the circle whose radius is the chord of the generating arc.*

In § 773, the area generated by arc $AB = \overline{AH} \cdot 2\pi r = \pi AH \cdot AF$.
But $AH \cdot AF = \overline{AB}^2$ (§ 394).

779. A **lune** is a portion of a spherical surface enclosed between two great semicircles. The **angle of the lune** is measured by the dihedral angle formed by the planes of the semicircles.

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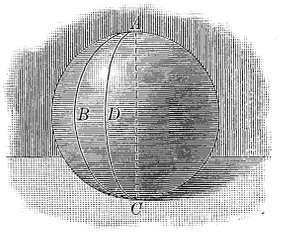
780. *A lune is measured by its angle.* For two lunes on a sphere having equal angles are equal (why?), and if the angle of one lune is any multiple of the angle of a second lune, the area of the first is the same multiple of the area of the second. (Cf. § 563.)



781. Corollary 5. *The area of a lune is* $\frac{n}{360} \cdot 4\pi r^2$, *where* n *is the number of degrees in the angle of the lune, and* r *is the radius of the sphere.*

EXERCISES

1. Find the area of a sphere whose radius is 10.

2. Find the radius of a sphere whose area is 154 (take $\pi = \frac{22}{7}$).

3. Derive a formula for the radius of a sphere in terms of its area.

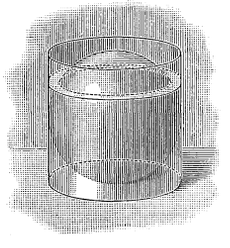
4. What is the radius of a sphere if its area has the same numerical value as the circumference of a great circle?

5. What is the area of a lune of 20° on a sphere whose radius is 6 in.?

6. How many degrees in the angle of a lune whose area is equal to that of a great circle of the sphere?

7. What is the ratio of the areas of two lunes, each of 48°, on spheres whose radii are 2 and 3 in.?

8. Find the area of the earth's surface, assuming it to be a perfect sphere of radius 4000 mi.?



9. Show that the area of a sphere is equal to the lateral area of the circumscribed cylinder of revolution.

10. Show that the area of a zone is equal to the portion of the lateral area of the circumscribed cylinder of revolution which is intercepted by the planes of the bases of the zone, the axis of the cylinder being perpendicular to the planes of the bases of the zone.

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11. Find the area of a zone of altitude 5 on a sphere of radius 15.

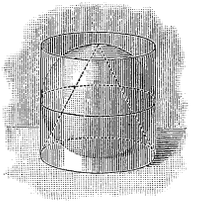
12. What portion of the surface of the earth is included between the parallels 30° N. Lat. and 30° S. Lat.?

13. What fraction of the whole spherical surface is illuminated from a point of light whose distance from the sphere is equal to the radius? the diameter? four times the radius? n times the radius?

14. What is the ratio of the areas of two zones of the same altitude on spheres of radii 5 and 12?

15. Show that the area of a lune is equal to that portion of the lateral area of the circumscribed cylinder of revolution which is intercepted by the planes of the sides of the lune, if these planes intersect in the axis of the cylinder.

16. Find the area on the earth's surface bounded by the equator, the parallel 30° N. Lat., and the meridians 40° and 120° W. Long. Take $r = 4000$ mi.



17. A right cylinder is circumscribed about a sphere, and a right circular cone is inscribed in the cylinder, having the same base and altitude as the cylinder. Find the ratio of the total areas of the cone, sphere, and cylinder.

18. A cone of revolution is circumscribed about a sphere. The vertex angle of a section of the cone through its axis is 60° (45°). Find the ratio of the total area of the cone and the area of the sphere, if the diameter of the base of the cone is 10 in.

782. Volume of a Sphere. Informal Discussion. The spherical surface, being a closed surface, encloses a definite volume. This volume may be found by various methods of approximation, two of which are considered in this and in § 786.

For a first method consider the volume of a sphere as the sum of the volumes of a large number of pyramids whose bases are small portions of the spherical surface (such as arise, for example, by drawing meridians and parallels for every minute

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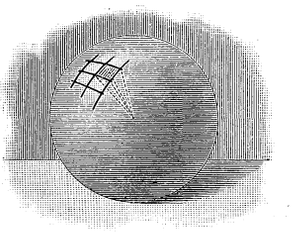
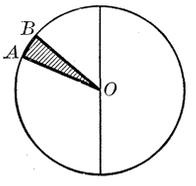
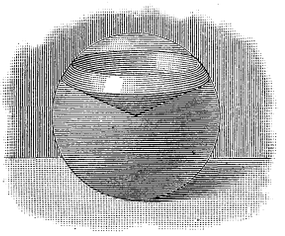
of longitude and latitude), so small that they may be regarded practically as plane figures, and each of whose altitudes is equal to the radius of the sphere. Since the volume of each pyramid equals $\frac{1}{3}bh$, their sum will be equal to $\frac{1}{3}r$ times the sum of the bases, or $\frac{1}{3}r \cdot 4\pi r^2$, which equals $\frac{4}{3}\pi r^3$.



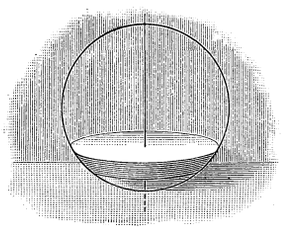
783. This treatment, however, does not admit of rigorous proof. For this reason it is desirable to use the method of § 786.

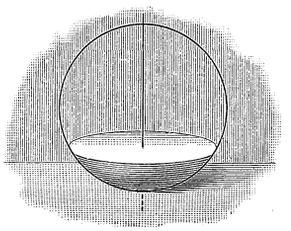
784. A **spherical sector** is a solid generated by the revolution of a sector of a circle about some diameter of the circle as an axis.





785. A **spherical segment** is a portion of a sphere included between two parallel planes. The circular sections formed on these planes are called the **bases** of the segment. If one of the planes is tangent to the sphere, the spherical segment is called a **segment of one base.**





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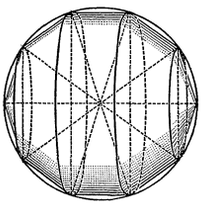
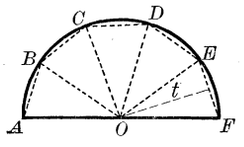
PROPOSITION VIII. THEOREM

786. *The volume of a sphere is equal to one third the product of the radius and the spherical surface.*





Given a sphere whose center is *O*, generated by the semicircle *ABCDEF* revolving about the diameter *AF* as an axis. Let *s* be the area of its surface, *r* the radius, and *v* its volume.

To prove *that* $v = \frac{1}{3}rs$.

Proof (Informal). 1. Inscribe in the semicircle half of a regular polygon having an even number of sides, as *ABCDEF*. Draw the apothems of this semipolygon and let t denote the value of each of them. Draw the radii to the vertices.

2. Then the volume generated by each triangle of the polygon is equal to $\frac{1}{3}t$ times the area generated by its base. §§ 721, 722

$\therefore$ the volume generated by the semipolygon is equal to $\frac{1}{3}t$ times the area generated by the semiperimeter.

3. Let the number of sides of the regular inscribed polygon be increased indefinitely. Then it may be assumed

that the area generated by the semiperimeter approaches s;

that the volume generated by the semipolygon approaches v;

and that the apothem t approaches r.

4. $\therefore v = \frac{1}{3}rs$.

787. COROLLARY 1. *The volume of a sphere equals* $\frac{4}{3}\pi r^3$ *or* $\frac{1}{6}\pi d^3$, *where r is the radius and d the diameter of the sphere.*

788. COROLLARY 2. *The volumes of two spheres are to each other as the cubes of their radii or of their diameters.*

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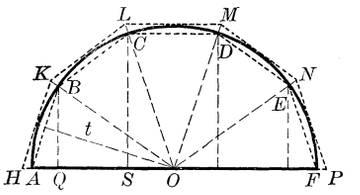
789. Corollary 3. *The volume of a spherical sector is equal to one third the product of the area of the zone which forms its base and the radius of the sphere.*

790. Area and Volume of a Sphere. Formal Demonstration. Suppose that the sphere is generated by the revolution of a semicircle about its diameter as an axis. If now half of a regular polygon of an even number of sides is inscribed in the semicircle, the sides of this semipolygon $ABCDEF$ will generate the lateral surfaces of a series of consecutive cones and frustums. Similarly, another such series will be generated by the sides of half of a similar regular circumscribed polygon $HKLMNP$.



'Now the area generated by $ABCDEF = 2\pi t \cdot AF$, § 720

t being the apothem.

Also, the area generated by $HKLMNP = 2\pi r \cdot HP$.

Let the number of sides of the inscribed polygon be indefinitely increased. Then the sum of the lateral areas of the series of cones and frustums will be an increasing variable which constantly remains less than the total area generated by $HKLMNP$.

For as the number of sides of the inscribed semipolygon is increased, t increases, while AF remains constant.

Also, no matter how many sides the inscribed polygon has,

$$t < r \text{ and } AF < HP. \quad \therefore t \cdot AF < r \cdot HP,$$

and
$$2\pi t \cdot AF < 2\pi r \cdot HP.$$

That is, the area generated by the inscribed semipolygon is an increasing variable approaching a limit as the number of sides is indefinitely increased (§ 482).

In like manner, the area generated by the circumscribed semipolygon as the number of sides is indefinitely increased can be shown to be a decreasing variable approaching a limit (§ 483), since r remains constant while HP grows less, but is always $> AF$.

It can now be shown that the areas generated by the inscribed and circumscribed semipolygons, respectively, approach the same limit; that is, that their ratio approaches *unity* as a limit. This ratio is

$$\frac{2\pi t \cdot AF}{2\pi r \cdot HP}, \text{ which equals } \frac{t \cdot AF}{r \cdot HP}.$$

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But in the similar triangles AOB and HOK,

$$t : r = AO : HO.$$

Whence $$t : r = AF : HP.$$

The preceding ratio thus becomes $\frac{t^2}{r^2}$.

Now r is constant, and $\frac{t}{r} \doteq 1$. (§ 485.) $\therefore \frac{t^2}{r^2} \doteq 1$.

That is, the ratio of the areas approaches *unity* as a limit, and hence the areas have a common limit.

It has also been shown (§ 786) that the volume generated by $ABCDEF = \frac{1}{3}t \cdot$ area generated by $ABCDEF$.

Similarly, the volume generated by $HKLMNP = \frac{1}{3}r \cdot$ area generated by $HKLMNP$.

Then it is easily proved that the volume generated by $ABCDEF$ is an increasing variable approaching a limit. In like manner, the volume generated by $HKLMNP$ is a decreasing variable approaching a limit. Further, the volumes generated by the inscribed and circumscribed polygons have a common limit; that is, their ratio approaches *unity* as a limit.

For this ratio is $$\frac{\frac{1}{3}t \cdot 2\pi t \cdot AF}{\frac{1}{3}r \cdot 2\pi r \cdot HP} = \frac{t^2 \cdot AF}{r^2 \cdot HP}.$$

But since $$\frac{t}{r} = \frac{AF}{HP} \text{(above)},$$

the ratio becomes $$\frac{t^3}{r^3}.$$

And, as before, this ratio approaches *unity* as a limit, and the volumes have a common limit. These considerations justify the

Definition. The **area of a sphere** is the common limit approached by the areas generated by the revolution about a diameter of half the perimeters of two regular polygons, one inscribed in a great circle of the sphere and the other circumscribed about the same great circle of the sphere, as the number of sides of these polygons is indefinitely increased; and the **volume of a sphere** is the common limit approached by the volumes generated by the revolution of these semipolygons.

Therefore the area, considering the inscribed polygon only, equals

$$2\pi r \cdot AF = 2\pi r \cdot 2r = 4\pi r^2,$$

while the volume equals

$$\frac{1}{3}r \cdot 2\pi r \cdot AF = \frac{1}{3}r \cdot 2\pi r \cdot 2r = \frac{4}{3}\pi r^3.$$

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EXERCISES

1. Find the volume of a sphere whose radius is 10.

2. Find the radius of a sphere whose volume is 38,808. (Take $\pi = \frac{22}{7}$.)

3. Given v the volume, s the area, and r the radius of a sphere, derive formulas for r and s in terms of v, and for r and v in terms of s.

4. Plot on one pair of axes the graphs of a, the area of a great circle; of s, the area of a sphere; and of v, the volume of the sphere, taking values of the radius as abscissas, and the corresponding values of a, s, and v, respectively, as ordinates. Give r values from 1 to 5, and take ten vertical divisions equal to one division of the horizontal scale. From the graph complete the following table:

r	1.5	2.7	3.1	3.6	4.2	4.5	4.9
a							
s							
v							

5. How many spherical shot $\frac{1}{8}$ in. in diameter can be made from a cylindrical bar of lead 10 in. long and $\frac{3}{4}$ in. in diameter?

6. If the diameter of a sphere is doubled, in what ratios are the surface and the volume increased?

7. It is desired to double the capacity of a balloon. By what number must the area be multiplied?

8. What is the weight of a spherical shell of iron if its external diameter is 6 in. and its thickness is $\frac{1}{8}$ in., given that a cubic inch of iron weighs .26 lb.?

9. How many iron balls 3 in. in diameter can be made from one 12 in. in diameter, there being no allowance for waste?

10. How many bullets of 32 calibre (.32 of an inch in diameter) can be cast from a piece of lead pipe 10 in. long, 1 in. in exterior diameter, and $\frac{1}{8}$ in. thick, there being no allowance for waste?

11. If the mean diameter of the earth is 7918 mi. and that of the moon 2162 mi., what is the approximate ratio of their volumes?

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12. The volumes of two spheres have the ratio 27 : 343. The sum of the radii is 5. Find the radii.

13. Find the volume of a sphere circumscribed about a regular tetrahedron of edge 12. (See Ex. 45, p. 335.)

14. Find the ratio of the volume of a sphere to that of the inscribed cube; to that of the circumscribed cube.

15. A right circular cylinder is circumscribed about a sphere 10 in. in diameter, and a right circular cone is inscribed in the cylinder, its base and altitude being equal to those of the cylinder. Show that the volume of the sphere is equal to the difference of the volumes of the cylinder and the cone.

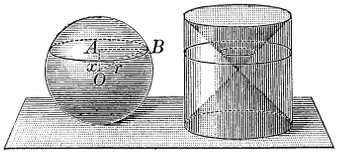
16. A rifle shell has the shape of a cylinder surmounted by a hemispherical cap. The total length of the shell is four times its diameter. Compare the surfaces and also the volumes of the cylindrical and the spherical portions.

17. A solid has the form of a right circular cone standing on the flat base of a hemisphere of equal radius. The cone and the hemisphere have equal volumes. Compare the surfaces of the conical and spherical portions of the solid.

18. A square, its inscribed circle, and its circumscribed circle are revolved about a diagonal of the square. Compare the volumes of the three solids generated; compare their areas.

19. The sum of the radii of two spheres is 4 in. The difference of their surfaces is 32π sq. in. Find the sum of their volumes.

20. Given a sphere 10 in. in diameter. A regular square pyramid is inscribed and its altitude is 8 in. Find the volume and total surface of the pyramid, and find their respective ratios to the volume and surface of the sphere.



21. Prove the formula for the volume of a sphere by Cavalieri's principle (see p. 430) by showing that a cross section of the sphere is constantly equal to that of the solid between the circumscribed cylinder of revolution and a double cone inscribed in the cylinder To simplify the figure, the sphere is placed beside the cylinder.

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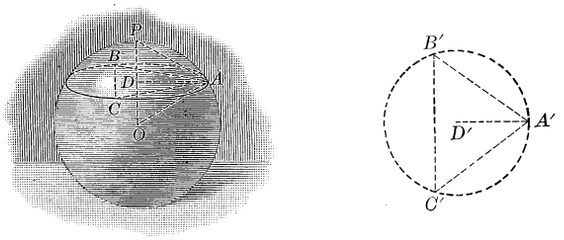
791. Geometry on a Spherical Surface. In the geometry on a spherical surface it is assumed that all constructions are made upon the surface of the sphere. In the proofs of these constructions it is often convenient, however, to use points within the sphere. The methods employed are in some cases identical with those of plane geometry. There are, nevertheless, many important differences. Some terms and figures employed in plane geometry have no corresponding terms and figures in spherical geometry, and vice versa (§ 838). To the straight line in plane geometry corresponds the arc of a great circle in spherical geometry. To construct a great circle of a sphere the radius of the sphere must first be found (§ 755). In many cases the radius cannot be obtained directly, but must be determined by measurements taken on the surface of the sphere. This is the case when the sphere is a material solid.

PROPOSITION IX. PROBLEM

792. *To find the radius of a material sphere.*



Construction. Place one foot of the compasses at any point P on the sphere, and with any convenient opening of the compasses, as PA, draw the circle ABC. Measure with the compasses the three chords AB, BC, and CA, and in any plane construct $\triangle A'B'C'$, having its sides respectively equal to AB, BC, and CA. Construct $D'A'$, the radius of the circumscribed circle of the triangle $A'B'C'$.

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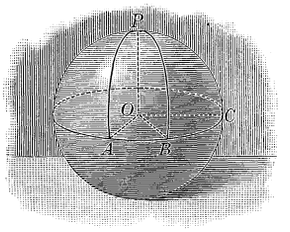
Now the chord PA can be taken as the base, and the radius $D'A'$ of the small circle as the altitude upon one leg of an isosceles triangle whose legs are radii of the sphere.

(Prove this construction.)

To draw the arc of a great circle through two given points the following theorem is employed.

PROPOSITION X. THEOREM

793. *A point on a spherical surface which is at a quadrant's distance from each of two other points not the ends of a diameter is the pole of a great circle passing through those two points.*



Given on the sphere whose center is *O* the point *P* at a quadrant's distance from *A* and from *B* on the great circle *ABC*.

To prove *that P is the pole of the great circle ABC.*

Proof. 1.	Draw OP, OA, and OB.	
2. Then	$PO \perp OA$ and OB.	Why?
3. Hence	$PO \perp$ plane ABC.	Why?
	$\therefore P$ is the pole of $\odot ABC$.	Def.

794. COROLLARY 1. *To draw through two given points on a spherical surface the arc of a great circle, describe arcs of great circles about the two given points as poles intersecting at a third point. This is the pole of the required great circle.*

795. COROLLARY 2. *If the pole of one great circle lies on another great circle, the planes of the circles are perpendicular to each other.*

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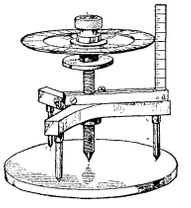
EXERCISES

1. Prove that the chord of the polar distance of a small circle on a sphere is a mean proportional between a diameter of the sphere and the distance of the pole from the plane of its circle (§ 394).

2. On a sphere, with a chord of 425 mm., a small circle is drawn, and the sides of an inscribed triangle are found to be respectively 264 mm., 600 mm., and 720 mm. Find the radius of the sphere.

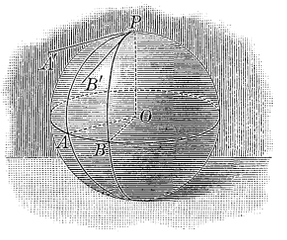
3. The **spherometer** is an instrument used to determine the radius of a sphere, or the radius of curvature of a spherical lens. The three feet of the tripod determine a small circle of known radius, and the height of the point of the screw above the plane of this circle is found by readings from the upright bar and the horizontal disk. The radius of curvature can now be determined (§ 393). Explain.



4. Find the radius of curvature of a spherical lens from the following data:

Radius of base (cm.)	1.5	2.5	2.	3.5	2.5
Height of point (cm.)	.146	.202	.284	.877	.126

796. A **spherical angle** is the figure formed when two arcs of great circles are drawn from the same point on the surface of a sphere. This point is called the **vertex,** and the arcs are called the **sides** of the spherical angle. A spherical angle is measured by the plane angle formed by the tangents to the sides at the vertex.



797. A diameter drawn from the vertex of a spherical angle is the intersection of the planes of its sides (§ 738). The tangents to the sides at the vertex are perpendicular to this diameter and lie in the planes of the sides. It follows that

A spherical angle has the same measure as the dihedral angle formed by the planes of its sides.

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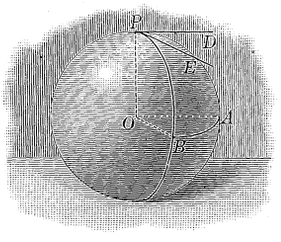
PROPOSITION XI. THEOREM

798. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole, and included between its sides, produced if necessary.*



Given the spherical angle *APB* formed by the great circles *PA* and *PB*; and *AB* the included arc of a great circle described about *P* as a pole.

To prove *that sph.$\angle APB$ is measured by arc AB.*

Proof. 1. Draw the radii OA and OB; draw PD and PE tangent to PA and PB respectively. Draw the radius PO.

2. Now $\angle POA$ is measured by the quadrant PA.

Hence $AO \perp OP$.

3. But $DP \perp OP$,

and AO and DP are in the plane POA.

4. Hence $AO \parallel DP$. Why?

5. In like manner, $BO \parallel EP$.

$\therefore \angle DPE = \angle AOB$. Why?

6. But $\angle AOB$ is measured by arc AB.

Hence the spherical angle APB is measured by arc AB.

799. A **right spherical angle** is formed when the planes of the sides of the angle are perpendicular to each other (§ 795). In this case, each side of the spherical angle is said to be **perpendicular** to the other.

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SPHERICAL POLYGONS AND POLYHEDRAL ANGLES

800. A **spherical polygon** is a figure on the surface of a sphere bounded by three or more arcs of great circles. The arcs are called the **sides,** and the spherical angles formed by the sides, the **angles** of the spherical polygon.

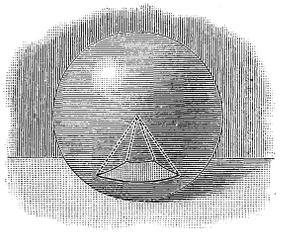
801. A **spherical triangle** is a spherical polygon of three sides.

Only spherical polygons each of whose sides is less than a semicircumference are considered in this text.

802. A **polyhedral angle** is the figure formed by three or more rays having a common origin, no three of the rays lying in one plane. The rays are called the **edges;** the common origin is called the **vertex;** the planes determined by consecutive edges are called the **faces;** the dihedral angles between consecutive faces, the **dihedral angles;** and the angles between consecutive edges, the **face angles,** of the polyhedral angle.

A polyhedral angle is called **trihedral, tetrahedral,** etc., according as it is formed by three rays, four rays, etc.

803. Spherical Polygon and Polyhedral Angle. From the definition of the spherical polygon it follows that the radii to the vertices of the polygon determine a polyhedral angle at the center of the sphere. This polyhedral angle is called the **corresponding central polyhedral angle.** Many of the properties of spherical polygons are derived from the properties of polyhedral angles, and conversely.



804. *The sides of a spherical polygon measure the face angles of the corresponding central polyhedral angle.* (Why?)

805. *The angles of a spherical polygon have the same measure as the dihedral angles of the corresponding central polyhedral angle* (§ 797).

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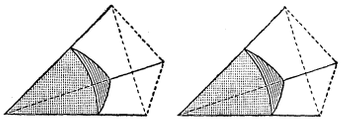
806. It should be noted that both sides and angles of a spherical polygon are measured in degrees. The lengths in linear units of the sides of a spherical polygon, however, depend upon the radius of the sphere.

807. A **convex polyhedral angle** is one no face of which, when produced, will enter the polyhedral angle.

808. A **convex spherical polygon** is one no side of which, when produced, will enter the polygon.

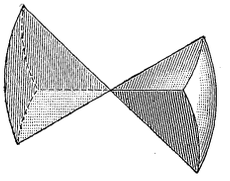
809. Two polyhedral angles are **congruent** if the face angles and dihedral angles of one polyhedral angle are respectively equal to those of the other and arranged in the same order.

For they can be made to coincide.



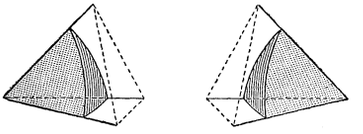
810. Two spherical polygons, on the same sphere or on equal spheres, are **congruent** if the sides and angles of one are respectively equal to those of the other and arranged in the same order.

811. Two polyhedral angles are **symmetric** if the face angles and dihedral angles of one polyhedral angle are respectively equal to those of the other, but arranged in reverse order.



When all the edges of a polyhedral angle are produced through the vertex, a symmetric polyhedral angle is formed.

812. Two spherical polygons, on the same sphere or on equal spheres, are **symmetric** if the sides and angles of one are respectively equal to those of the other, but arranged in reverse order.



When diameters are drawn from each vertex of a spherical polygon, their other extremities are the vertices of a symmetric spherical polygon.

In general, symmetric polyhedral angles or spherical polygons cannot be made to coincide.

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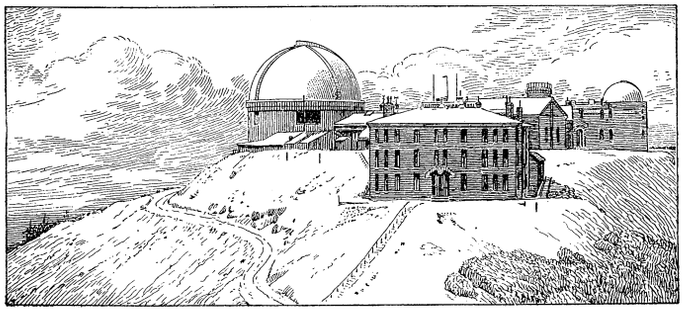
813. Homologous parts (face angles and dihedral angles) of congruent or symmetric polyhedral angles are equal.

814. Homologous parts (sides and angles) of congruent or symmetric spherical polygons are equal.

NOTE. The following paragraphs will develop some of the properties of spherical triangles. These occupy very much the same place in spherical geometry as plane triangles in plane geometry. It was found in the case of plane triangles that, given certain determining parts of a triangle, the remaining parts could be obtained either by construction or by measurement. This fact is of fundamental importance in constructions, in surveying, and in trigonometry (Plane Geometry, pp. 61–68, 181–184, 259–264). Similar statements hold with reference to spherical triangles.



The theorems on spherical triangles are used extensively in navigation and astronomy. In fact, the whole science of spherical triangles is an outgrowth of astronomy. From time immemorial people observed the regular march of the stars across the celestial sphere (see Ex. 11, p. 444). Gradually these observations, at first very crude, were recorded in tables and almanacs. Astronomical records dating back to nearly 2000 B.C. have been discovered. The growth of exact and scientific methods was very slow. The first known treatise on spherical triangles is due to Menelaus of Alexandria (98 A.D.), while it was the astronomer Ptolemy of Alexandria (150 A.D.) who first made use of these triangles for purposes of computation.

The making and recording of astronomical observations is carried on in observatories, of which the famous Lick Observatory, at Mt. Hamilton, California, shown in the picture, is one of the finest.

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Preliminary Theorems

815. *Two trihedral angles are congruent if they have a face angle and the two adjoining dihedral angles of one equal respectively to a face angle and the two adjoining dihedral angles of the other and arranged in the same order, and symmetric if the given parts of one are arranged in reverse order to those of the other.*

Prove by superposition that one is congruent to the other if the parts are arranged in the same order. If the parts are not thus arranged, construct a trihedral angle (§ 811) which is symmetric to one of those given, and proceed as before.

816. *Two spherical triangles, on the same sphere or on equal spheres, are congruent if they have a side and the two adjoining angles of one equal respectively to a side and the two adjoining angles of the other and arranged in the same order, and symmetric if the given parts of one are arranged in reverse order to those of the other.*

Construct the corresponding central polyhedral angles and use the preceding theorem and §§ 804–805.

817. *Two trihedral angles are congruent if they have two face angles and the included dihedral angle of one equal respectively to two face angles and the included dihedral angle of the other and arranged in the same order, and symmetric if the given parts of one are arranged in reverse order to those of the other.*

Proceed as in § 815.

818. *Two spherical triangles, on the same sphere or on equal spheres, are congruent if they have two sides and the included angle of one equal respectively to two sides and the included angle of the other and arranged in the same order, and symmetric if the given parts of one are arranged in reverse order to those of the other.*

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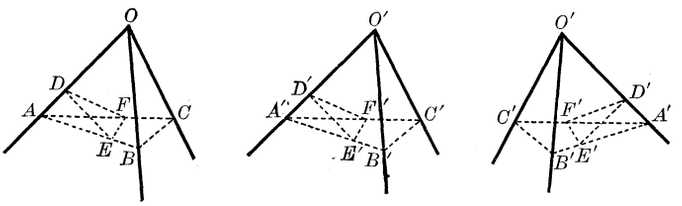
PROPOSITION XII. THEOREM

819. *Two trihedral angles are congruent if the three face angles of the one are respectively equal to the three face angles of the other and all are arranged in the same order, and symmetric if the parts of one are arranged in reverse order to those of the other.*



Given in the trihedral angles *O-ABC* and *O′-A′B′C′*, angle *AOB* equal to angle *A′O′B′*, angle *BOC* equal to angle *B′O′C′*, and angle *COA* equal to angle *C′O′A′*.

To prove *that* $O\text{-}ABC \equiv O'\text{-}A'B'C'$, *or* $O\text{-}ABC$ *is symmetric to* $O'\text{-}A'B'C'$.

Proof. 1. On the six edges lay off equal segments OA, OB, OC, $O'A'$, $O'B'$, and $O'C'$. Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

2. Then $\triangle AOB \equiv \triangle A'O'B'$, Why?

and $AB = A'B'$.

3. In like manner $BC = B'C'$,

and $CA = C'A'$.

4. On AO and $A'O'$ respectively lay off AD equal to $A'D'$. Draw DE in face AOB, and DF in face AOC, each $\perp$ to OA. These lines meet AB and AC respectively since $\triangle AOB$ and $\triangle AOC$ are isosceles. In like manner draw $D'E'$ in face $A'O'B'$, and $D'F'$ in face $A'O'C'$, each $\perp$ to $O'A'$. Draw EF and $E'F'$.

5. (Outline) Now $\triangle ADE \equiv \triangle A'D'E'$. Why?

6. And $\triangle ADF \equiv \triangle A'D'F'$. Why?

7. Also $\triangle ABC \equiv \triangle A'B'C'$. Why?

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8. $\therefore \triangle AEF \equiv \triangle A'E'F'$. Why?

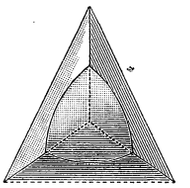
9. $\therefore \triangle DEF \equiv \triangle D'E'F'$. Why?

10. $\therefore$ dihedral angle B-OA-C = dihedral angle B'-$O'A'$-C'.

(Being measured by equal plane angles.)

11. $\therefore$ trihedral angle O-ABC and trihedral angle O'-$A'B'C'$ are congruent if the parts are arranged in the same order, or symmetric if the parts are arranged in reverse order. § 817

820. Corollary. *Two spherical triangles, on the same sphere or on equal spheres, are congruent if they have three sides of one equal respectively to three sides of the other and arranged in the same order, and symmetric if the sides of one are arranged in reverse order to those of the other.*



821. A trihedral angle is called **isosceles** if it has two face angles equal.

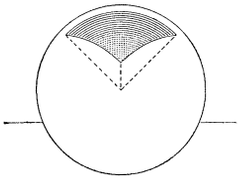
822. A spherical triangle is called **isosceles** if it has two sides equal.

Preliminary Theorems

823. *If a trihedral angle is isosceles, the dihedral angles opposite the equal face angles are equal.*

Construct a symmetric trihedral angle (§ 811). This trihedral angle will now be congruent to the original trihedral angle (§ 817).

824. *If a spherical triangle is isosceles, the angles opposite the equal sides are equal.*



825. *If two dihedral angles of a trihedral angle are equal, the face angles opposite are equal.*

826. *If two angles of a spherical triangle are equal the sides opposite are equal.*

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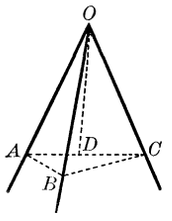
PROPOSITION XIII. THEOREM

827. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral angle *O-ABC*, and *AOC* its greatest face angle.

To prove *that* $\angle AOB + \angle BOC > \angle AOC$.

Proof. 1. In the face AOC, construct $\angle AOD = \angle AOB$, and take OD equal to OB. Draw AB, AC, and BC.

2. Then $\triangle AOB \equiv \triangle AOD$, Why?

and $AB = AD$.

3. But $AB + BC > AC$. Why?

Subtracting the equals AB and AD,

$BC > DC$.

4. But in the $\triangle BOC$ and DOC,

OC is common,

$OD = OB$, Cons.

and $BC > DC$.

Hence $\angle BOC > \angle DOC$. § 232

5. Adding the equal angles AOB and AOD,

$\angle AOB + \angle BOC > \angle AOD + \angle DOC$,

or $\angle AOB + \angle BOC > \angle AOC$.

828. COROLLARY. *The sum of two sides of a spherical triangle is greater than the third side.*

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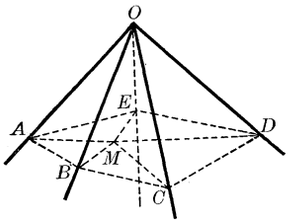
PROPOSITION XIV. THEOREM

829. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given the convex polyhedral angle *O-ABCDE*.

To prove *that the sum of the face angles is less than* 4 *rt.* ∡s.

Proof. 1. Pass a plane intersecting the edges in $A, B, C, \cdots$ and the faces in AB, BC, $CD, \cdots$. Join any point M in this plane to A, B, $C, \cdots$.

2. Then $\angle OBA + \angle OBC > \angle ABC$, § 827

and $\angle OCB + \angle OCD > \angle BCD$,

etc.

Add these inequalities.

Then the sum of all the base angles of the triangles whose common vertex is O is greater than the sum of all the base angles of the triangles whose common vertex is M.

3. But the sum of *all* the angles of the triangles whose common vertex is O is equal to the sum of *all* the angles of the triangles whose common vertex is M. Why?

4. Hence the sum of all the angles at O is less than the sum of all the angles at M. § 222, Ax. 11

5. But the sum of all the angles at $M = 4$ rt. ∡s.

∴ the sum of the face angles at $O < 4$ rt. ∡s.

830. COROLLARY. *The sum of the sides of any convex spherical polygon is less than 360°.*

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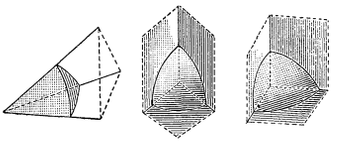
831. A trihedral angle is called a **rectangular,** a **birectangular,** or a **trirectangular trihedral angle,** according as it has one, two, or three right dihedral angles.

For example, three concurrent edges of a cube form a trirectangular trihedral angle, while three concurrent edges of a regular triangular prism form a birectangular trihedral angle.

832. A spherical triangle is called a **right,** a **birectangular,** or a **trirectangular spherical triangle,** according as it has one, two, or three right spherical angles.



Preliminary Theorems

833. *In a birectangular trihedral angle, the face angles opposite the right dihedrals are right angles* (§ 572); *and conversely, if two face angles of a trihedral angle are right angles, the trihedral angle is birectangular* (§ 566).

834. *In a birectangular spherical triangle the sides opposite the equal angles are quadrants; and, conversely, if two sides of a spherical triangle are quadrants, the triangle is birectangular.*

835. *In a trirectangular trihedral angle the face angles are right angles, and conversely.*

836. *In a trirectangular spherical triangle the sides are quadrants, and conversely.*

837. *Three mutually perpendicular planes passed through the center of a sphere divide the spherical surface into eight congruent trirectangular spherical triangles. All trirectangular spherical triangles on the same sphere are congruent. The area of a trirectangular spherical triangle is equal to one eighth of the whole spherical surface.*

838. Other theorems and constructions are established by the methods of plane geometry. The following table shows

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the corresponding figures in spherical geometry and plane geometry. Note that the correspondence is not complete.

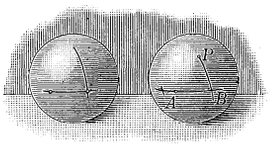
Spherical Geometry	Plane Geometry
Arc of great circle	Straight line
Small circle	Circle
Spherical angle	Angle
Perpendicular arcs	Perpendicular lines
Pole of small circle	Center of circle
Pole of great circle	.....
.....	Parallel lines
.....	Similar figures
Spherical triangle	Triangle
Right spherical triangle	Right triangle
Isosceles spherical triangle	Isosceles triangle
Equilateral spherical triangle	Equilateral triangle
Birectangular spherical triangle	.....
Trirectangular spherical triangle	.....
Spherical polygon	Polygon
Lune	.....
Congruent spherical triangles Symmetrical spherical triangles	Congruent triangles

EXERCISES

In the following, prove the theorems, and solve and prove the problems:

1. Through a given point to draw an arc of a great circle perpendicular to a given arc.

Suggestion. Take on the given arc a point which is a quadrant's distance from the given point. This is the pole of a great circle which is perpendicular to the given arc (§ 795).



2. To bisect a given arc of a great circle.

Suggestion. About the ends of the given arcs as poles, with equal polar distances sufficiently great, describe small circles intersecting in two points. Join these points by a great circle arc, which will be the perpendicular bisector of the given arc. The proof follows the method of Book I, Proposition VI.

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3. To bisect a given spherical angle.

4. Through a given point *not the pole of a given great circle arc* only one great circle arc can be drawn perpendicular to the given arc. (§§ 739, 740. The proof follows § 161.)

5. If two right, *but not birectangular*, spherical triangles on a sphere have the side opposite the right angle and another side in one respectively equal to the corresponding parts in the other, the spherical triangles are congruent or symmetric. (Cf. § 165.)

6. Every point in the perpendicular bisector of a great circle arc is equidistant from the ends of the arc; and, conversely, every point equidistant from the ends of a great circle arc lies on the perpendicular bisector.

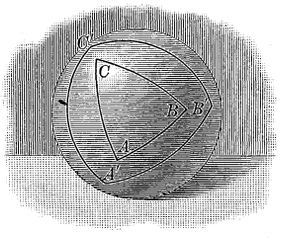
7. To circumscribe a small circle about a given spherical triangle.

8. To find the pole of a given small circle.

9. Small circles circumscribed about two congruent or symmetrical spherical triangles on the same sphere are equal.

10. Theorem. If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle; and conversely.

839. Polar Triangles. If three great circles are described about the vertices of a spherical triangle as poles, eight spherical triangles are formed. In one of these triangles it will be found that each vertex lies on the same side of the opposite side of the triangle as the given pole of that side; that is, as the corresponding vertex of the original triangle. This triangle is called the **polar triangle** of the first triangle.



Thus, of the eight triangles determined by the great circles drawn about A, B, and C as poles, the polar triangle $A'B'C'$ is the one which has A and A' on the same side of $B'C'$, B and B' on the same side of $A'C'$, and C and C' on the same side of $A'B'$.

Polar triangles are often used to prove propositions concerning spherical triangles generally.

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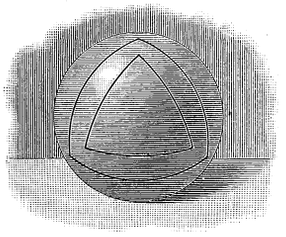
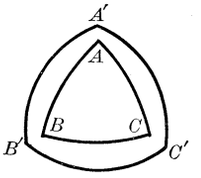
PROPOSITION XV. THEOREM

840. *If one spherical triangle is the polar triangle of another, then reciprocally the second spherical triangle is the polar triangle of the first.*





Given that the spherical triangle $A'B'C'$ is the polar triangle of the spherical triangle ABC.

To prove *that sph.* $\triangle ABC$ *is the polar triangle of sph.* $\triangle A'B'C'$.

Proof. 1. Since A is the pole of $B'C'$, the polar distance AB' is a quadrant. Hyp.

2. And since C is the pole of $A'B'$, the polar distance CB' is a quadrant.

3. $\therefore B'$ is the pole of AC. § 793

Similarly, A' is the pole of BC,

and C' is the pole of AB.

4. $\therefore$ sph. $\triangle ABC$ is the polar triangle of sph. $\triangle A'B'C'$. § 839

EXERCISES

1. In the preceding theorem give the proof assuming that sph. $\triangle ABC$ is the polar triangle of sph. $\triangle A'B'C'$.

2. Sketch the figure corresponding to the right-hand figure at the top of this page, assuming AB and AC to be each less than a quadrant, but BC greater than a quadrant; also assuming AB and AC to be each greater than a quadrant, but BC less than a quadrant.

3. Show that the polar triangle of a birectangular spherical triangle is also birectangular.

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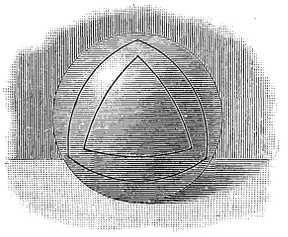
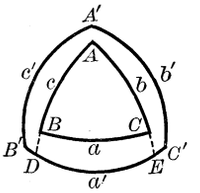
PROPOSITION XVI. THEOREM

841. *In two polar triangles, each angle of the one is measured by the supplement of the side opposite in the other.*





Given the polar triangles *ABC* and *A′B′C′*.

To prove *that* $\angle C$ *is measured by the supplement of* $A'B'$, $\angle B$ *by the supplement of* $C'A'$, *and* $\angle A$ *by the supplement of* $B'C'$.

Proof. 1. If necessary, produce the sides of $\angle A$ until they meet $B'C'$ in D and E respectively.

2. Then $\angle A$ is measured by arc DE. § 798

3. Also $B'E$ and DC' are each 90°, § 840

and $B'E + DC' = 180°$.

Hence $B'D + DE + DC' = 180°$.

That is, $B'D + DC' + DE = 180°$.

In other words, DE and $B'C'$ are supplementary.

4. $\therefore \angle A$ is measured by the supplement of $B'C'$.

In like manner, $\angle B$ is measured by the supplement of $C'A'$, and $\angle C$ is measured by the supplement of $A'B'$.

Similarly, the angles of the spherical triangle $A'B'C'$ are measured by the supplements of the corresponding opposite sides of $\triangle ABC$.

REMARK. If the letter at each vertex denotes the value of the angle in degrees, and small letters denote the sides in degrees, the theorem may be written

$$A + a' = 180°, \qquad B + b' = 180°, \qquad C + c' = 180°.$$

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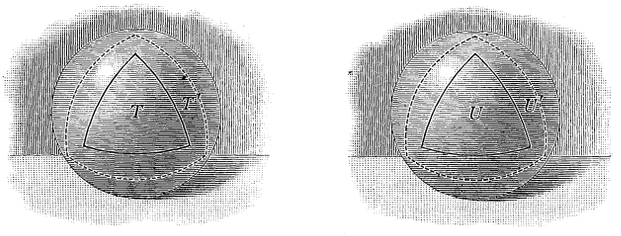
PROPOSITION XVII. THEOREM

842. *If two triangles on the same sphere or on equal spheres are mutually equiangular, they are mutually equilateral, and are either congruent or symmetric.*



Given that spherical triangles T and U are mutually equiangular.

To prove *that T and U are mutually equilateral, and are either congruent or symmetric.*

Proof. 1. Construct T', the polar triangle of T, and U', the polar triangle of U.

2. Since T and U are mutually equiangular (Hyp.),
T' and U' are mutually equilateral. § 841

3. $\therefore$ T' and U' are either congruent or symmetric. § 820

Hence T' and U' are mutually equiangular.

4. But T and U are the polar triangles of T' and U'. § 840

Hence T and U are mutually equilateral. § 841

$\therefore$ T and U are either congruent or symmetric. § 820

843. COROLLARY. *If two trihedral angles have their dihedral angles respectively equal, their face angles also are respectively equal, and they are either congruent or symmetric.*

Discussion. How many cases have now been considered in which it is proved that two spherical triangles are either congruent or symmetric? Are there congruence theorems for triangles in plane geometry corresponding to all of these cases?

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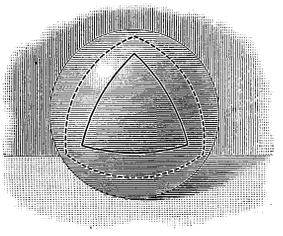
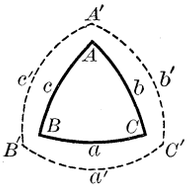
PROPOSITION XVIII. THEOREM

844. *The sum of the angles of a spherical triangle is greater than two and less than six right angles.*





Given the spherical triangle *ABC*, in which *A*, *B*, and *C* respectively denote the number of degrees in the three angles.

To prove *that* $A+B+C>180°$ *and* $<540°$.

Proof. 1. Construct the polar triangle $A'B'C'$, and denote the number of degrees in $B'C'$, $A'C'$, and $A'B'$ respectively by a', b', and c' respectively.

2. Then $A+a'=180°$. § 841

$B+b'=180°$.

$C+c'=180°$.

3. Adding these equations,

$$(A+B+C)+(a'+b'+c')=540°.$$

$$\therefore A+B+C<540°.$$

4. But $a'+b'+c'<360°$. § 830

$\therefore A+B+C>180°$. § 222, Ax. 11.

845. COROLLARY. *The sum of the dihedral angles of a trihedral angle is greater than two and less than six right angles.*

Discussion. What are the upper and lower limits for the sum of the angles of a spherical quadrilateral? of a spherical polygon of n sides?

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EXERCISES

1. The sides of a spherical triangle are 70°, 50°, and 100°. Find the angles of the polar triangle.

2. The angles of a spherical triangle are 30°, 80°, and 100°. Find the sides of the polar triangle.

3. Show that the polar triangle of an isosceles spherical triangle is also isosceles.

4. What is known of the polar triangle of an equilateral spherical triangle?

5. What spherical triangle is its own polar triangle?

846. Area of Spherical Figures. Areas of figures on the surface of a sphere (except zones) are not usually measured directly in units of plane area. Instead, a unit is adopted which is a portion of the spherical surface itself.

847. The **spherical degree** is equal to the area of a spherical triangle two of whose sides are quadrants, while the third is a great circle arc of one degree.

A spherical degree on the earth's surface, for example, is the area of a triangle two of whose sides are semimeridians, while the third is an equatorial arc of one degree. The spherical degree may also be defined as a semilune whose angle is one degree.

From the above definition it follows that

848. *The surface of a sphere is equal to 720 spherical degrees.*

849. *The number of spherical degrees in the area of a lune is twice the number of angular degrees in the angle of the lune.*

850. The **spherical excess** *of a spherical triangle* is the excess of the sum of the angles of the triangle over 180° (§ 844).

If a mariner is to lay out a course over a comparatively short distance by constructing a triangle, he may regard the triangle as a plane figure (plane sailing). But if the distance is longer, he must allow for the curvature of the earth, and the sum of the angles of the triangle will *exceed* 180°.

851. The **spherical excess** *of a spherical polygon of n sides* is the excess of the sum of its angles over $(n - 2)\ 180°$.

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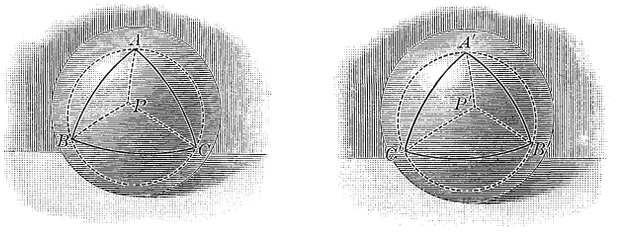
PROPOSITION XIX. THEOREM

852. *Two symmetric spherical triangles on the same sphere or equal spheres are equal in area.*



Given that spherical triangles *ABC* and *A'B'C'* are symmetric.

To prove *that* $sph.\triangle ABC = sph.\triangle A'B'C'$.

Proof. 1. Let P and P' be the poles of small circles passing through A, B, C, and A', B', C' respectively.

2. Then $\odot ABC = \odot A'B'C'$.

(Being circumscribed about congruent plane triangles.)

3. Draw the arcs of great circles PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$. These arcs are equal. § 753

4. Sph. $\triangle ABP$ and $A'B'P'$ are isosceles and symmetric.

Hence $\text{sph.}\triangle ABP \equiv \text{sph.}\triangle A'B'P'$. Why ?

In like manner, $\text{sph.}\triangle ACP \equiv \text{sph.}\triangle A'C'P'$,

and $\text{sph.}\triangle BCP \equiv \text{sph.}\triangle B'C'P'$.

5. Adding, area $\text{sph.}\triangle ABC = \text{area sph.}\triangle A'B'C'$.

If the two poles P and P' fell without their respective triangles, each triangle would be equal to the sum of two isosceles triangles diminished by a third isosceles triangle, the proof being unchanged otherwise.

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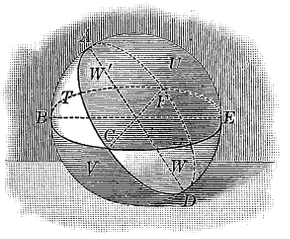
PROPOSITION XX. THEOREM

853. *The area of a spherical triangle measured in spherical degrees is equal to its spherical excess measured in angular degrees.*



Given the spherical triangle *ABC*, the letter at the vertex of each angle representing its value in degrees, and *T* the area in spherical degrees.

To prove $T = A + B + C - 180.$

Proof. 1. Complete the great circles of which the sides of the triangle are parts, giving the points of intersection D, E, F, and forming eight triangles, of which four, whose areas are T, U, V, and W, make up a hemispherical surface. Let W' be the area of the triangle ABF.

2. Then $\triangle CDE$ is symmetric to $\triangle ABF$. § 812

$\therefore W = W'$. § 852

3. Also $T + W' = 2\,C$. § 849

(Since the sph. $\measuredangle ABC$ and ABF form a lune, whose angle is C.)

$\therefore T + W = 2\,C$.

Also $T + U = 2\,B$,

and $T + V = 2\,A$.

4. Adding, $3\,T + U + V + W = 2\,A + 2\,B + 2\,C$.

5. But $T + U + V + W = 360$. § 848

Subtracting, $\therefore 2\,T = 2\,A + 2\,B + 2\,C - 360$.

Whence $T = A + B + C - 180$. Ax. 5

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854. COROLLARY 1. *The area of a spherical polygon measured in spherical degrees is equal to its spherical excess measured in angular degrees.*

For a spherical polygon of n sides may be broken up into $(n-2)$ spherical triangles, the area of each of which is equal numerically to its spherical excess. But the sum of the spherical excesses of the triangles is equal to the spherical excess of the polygon (§ 851).

855. COROLLARY 2. *The area of a spherical polygon equals*

$$\frac{e}{720} \cdot 4\pi r^2,$$

where e is the spherical excess of the polygon and r is the radius of the sphere.

856. A **spherical pyramid** is a solid bounded by a spherical polygon and the planes of its sides.

857. A **spherical wedge** is a solid bounded by a lune and the planes of its sides.

These solids may easily be shown to have the same ratio to the sphere that their bases have to the spherical surface. Since the volume of a sphere is equal to one third the product of its radius and its surface (§ 786), the volume of a spherical pyramid, or of a wedge, is one third the product of the radius of the sphere and the area of its base.

858. COROLLARY 3. *The volume of a spherical pyramid equals*

$$\frac{e}{720} \cdot \frac{4}{3}\pi r^3,$$

where e is the spherical excess of the base and r is the radius of the sphere.

EXERCISES

Find the areas of the following spherical triangles or polygons with the given angles, on spheres of the given radii. (*Take* $\pi = \frac{22}{7}$.)

1. 90°, 125°, 145°; $r = 7$ in.

2. 80°, 100°, 140°; $r = 10$ in.

3. 118°, 127°, 79°; $r = 14$ in.

4. 141°, 108°, 96°; $r = 20$ in.

5. 76° 31′, 94° 19′, 20° 10′; $r = 4000$ mi.

6. 40°, 85°, 125°, 130°; $r = 1$ ft.

7. 70°, 96°, 112°, 142°; $r = 7$ in.

8. 84°, 110°, 124°, 152°, 160°; $r = 14$ in.

9. 127° 30′, 140° 42′, 104° 37′, 110° 11′; $r = 10$ in.

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REVIEW EXERCISES

1. How many degrees in each angle of an equilateral spherical triangle of area 33 sq. in. on a sphere of radius 3.5 in.?

2. The sides of a spherical triangle are 64°, 71°, 85°. Find the area of its polar triangle if the radius of the sphere is 9 in.

3. Find the volume of a spherical pyramid if the angles of the base are 77°, 84°, and 109°, and the radius of the sphere is $10\frac{1}{2}$ in.

4. Show that the volume of a spherical segment of one base is $\pi h^2 (r - \frac{1}{3} h)$, if its altitude is h, and the radius of the sphere is r. (The spherical segment may be regarded as the difference between a spherical sector and a cone of revolution whose altitude is $r - h$ and the radius of whose base is $\sqrt{2rh - h^2}$.)

5. The radius of the base of a segment of one base is 24 ft., and the altitude of the segment is 12 ft. What is its volume?

6. A spherical segment of one base .8 in. high contains .3 cu. in. What is the radius of the sphere?

7. Find the formula for the volume of a spherical segment of one base in terms of the altitude of the segment and the radius of its base.

8. In a given sphere are inscribed a cone whose slant height is equal to the diameter of its base, and a cylinder whose altitude is equal to the diameter of its base. Prove that the volume of the cylinder is a mean proportional between the volumes of the sphere and cone.

9. Does the same relation hold in the case of the total areas of the figures mentioned in Ex. 8?

10. A spherical shell is bounded by two concentric spherical surfaces. Find a formula for the volume of metal in such a shell if its outside diameter is d, and its inside diameter is d', in terms of d and the thickness t of the shell. Show that the volume is approximately $\pi d^2 t$ if the thickness is small in proportion to the diameter.

11. The **specific gravity** of a homogeneous body floating in water is found by dividing the volume below the level of the water by the total volume. Find the specific gravity of a ball if it floats immersed to a depth equal to nine tenths of its diameter.

12. What is the specific gravity of a spherical body floating in water if one half of its surface is under water? one third? seven tenths?

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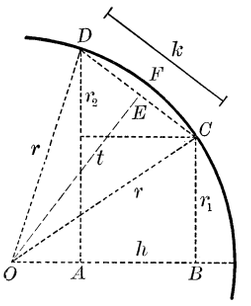
13. Derive a formula for the volume of a spherical segment of two bases. The spherical segment may be regarded as a conical frustum of revolution combined with the difference between a spherical sector and a solid of revolution. Consider the segment as generated by the revolution of the area $ABCFD$ about AB as an axis. Let O be the center of the sphere. Let r be the radius of the sphere, r_1 the radius of the upper base of the segment, r_2 the radius of the lower base, h the altitude of the segment, t the altitude of the triangle ODC, and k the slant height of the frustum. Then



vol. sph. seg. $ABCFD$
= vol. sph. sect. $OCFD$ − vol. solid of rev. $OCED$
+ vol. frustum $ABCED$.

Now vol. sph. sect. $OCFD = \frac{2}{3}\pi r^2 h$, § 789

and vol. solid of rev. $OCED = \frac{2}{3}\pi t^2 h$. § 722

$\therefore$ vol. sph. ring $DFCE = \frac{2}{3}\pi r^2 h - \frac{2}{3}\pi t^2 h = \frac{2}{3}\pi h(r^2 - t^2)$
$= \frac{2}{3}\pi h k^2/4 = \frac{1}{6}\pi h k^2 = \frac{1}{6}\pi h(\overline{r_2 - r_1}^2 + h^2)$.

Now vol. frustum $ABCD = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$. § 717

$\therefore$ vol. sph. seg. $ABCDF$
$= \frac{1}{6}\pi h(\overline{r_2 - r_1}^2 + h^2) + \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$
$= \frac{1}{6}\pi h(r_2^2 - 2r_1 r_2 + r_1^2 + h^2 + 2r_1^2 + 2r_1^2 + 2r_1 r_2)$
$= \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$.

This formula is often written $\frac{1}{2}\pi r_1^2 h + \frac{1}{2}\pi r_2^2 h + \frac{1}{6}\pi h^3$. That is,

The volume of a spherical segment is equal to the sum of the volumes of two cylinders and a sphere, the radii of the cylinders being the radii of the upper and lower bases respectively of the segment, their altitudes each one half the altitude of the segment, and the diameter of the sphere being the altitude of the segment.

14. Derive from the above formula a special formula for the volume of a segment of one base. Compare with the formula in Ex. 4.

15. Find the volume of a spherical segment if the radii of the bases are 6 and 10 and the altitude is 8.

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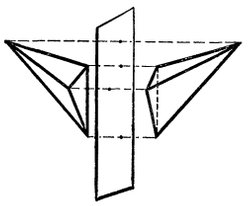
SUPPLEMENTARY PROPOSITIONS

Congruent, Symmetric, and Similar Solids

859. Two polyhedrons (and, in general, two solid figures) are **congruent** when they can by superposition be made to coincide (§ 113). **Homologous parts** (edges, face angles, and dihedral angles) of congruent polyhedrons are equal.

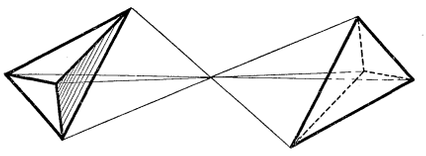
860. Plane Symmetry. Two polyhedrons (and, in general, two figures) are said to be *symmetric with respect to a plane* when to every point of the first polyhedron there corresponds a point of the second polyhedron such that the given plane, called the *plane of symmetry*, is perpendicular to the line segment joining the two corresponding points and bisects that segment.



The simplest example of symmetric solids arises when any object is compared with its reflection in a mirror. It will be noted that a polyhedron and its reflection are in general not congruent.

Numerous examples of symmetric solids occur in nature, as the right and left hands, the two shells of many forms of bivalves, etc.

861. Point Symmetry. Two polyhedrons (and, in general, two figures) are said to be *symmetric with respect to a point* when to every point of the first polyhedron there corresponds a point of the second such that the given point, called the *center of symmetry*, bisects the line joining these two corresponding points.



Symmetric polyhedral angles and spherical polygons (§§ 811, 812) are illustrations of point symmetry.

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862. Three-dimensional figures may also have *axial*, or *line symmetry*, which is identical in character with that of plane figures (Plane Geometry, §§ 96–101). Two solids symmetric with respect to a line may be made to coincide by revolving one of them about the line through half a complete revolution. Line symmetry is not further considered in this text.

863. It can be proved in respect to both point symmetry and plane symmetry that the figure symmetric to a straight line is a straight line, and that the figure symmetric to a plane surface is a plane surface; that the figure symmetric to a line segment is an equal line segment, and that the figure symmetric to a plane triangle is a congruent triangle (s. s. s.). From this it follows that the figure symmetric to a plane angle is an equal plane angle (§ 135), and therefore the figure symmetric to a dihedral angle is an equal dihedral angle (§ 563).

864. *The figure symmetric to a given polyhedron is a polyhedron whose parts are equal respectively to those of the given polyhedron, but are arranged in reverse order.*

865. *Two polyhedrons which are symmetric to the same polyhedron are congruent.*

Their parts are respectively equal (Ax. 1), and a double reversal of order of arrangement gives the same order.

866. A single polyhedron, or other solid figure, may be **symmetric** to itself with respect to a plane (or a point). In this case the plane is called a **plane of symmetry** (or the point a **center of symmetry**) of the figure.

EXERCISES

1. Show that a sphere has plane symmetry; also a cylinder of revolution; a cone of revolution; a right prism.

2. How many planes of symmetry has a rectangular parallelepiped?

3. Give illustrations of symmetry from architecture and decoration; from objects of common use.

4. Show that a sphere has a center of symmetry; a cylinder of revolution; a parallelepiped.

867. Congruence of Tetrahedrons. There are many different conditions under which two tetrahedrons can be proved congruent. Only one such congruence will be considered here.

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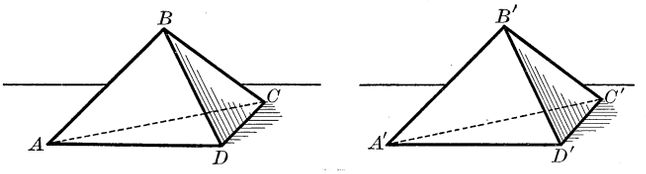
PROPOSITION I. THEOREM

868. *Two tetrahedrons are congruent if three faces of one are congruent respectively to three faces of the other and arranged in the same order.*



Given the tetrahedrons *ABCD* and *A′B′C′D′* with triangle *ABC* congruent to triangle *A′B′C′*, triangle *ABD* congruent to triangle *A′B′D′*, and triangle *ACD* congruent to triangle *A′C′D′* and arranged in the same order.

To prove *that tetrahedron* $ABCD \equiv$ *tetrahedron* $A'B'C'D'$.

Proof. 1. Trihedral $\angle A\text{-}BCD \equiv$ trihedral $\angle A'\text{-}B'C'D'$. § 819

Superpose these congruent trihedral angles.

Then B will fall on B', C on C', and D on D'. Why?

2. $\therefore \triangle BCD$ coincides with $\triangle B'C'D'$.

3. Hence the two tetrahedrons coincide and are congruent.

869. COROLLARY. *Two tetrahedrons have equal volumes if three faces of one are congruent respectively to three faces of the other, but arranged in reverse order.*

If P and Q are the given tetrahedrons, construct the tetrahedron P' symmetric to P with respect to one of the faces of P. Then P' and P have equal volumes (§ 712). Also $P' \equiv Q$ (§ 868). Hence P and Q have equal volumes.

EXERCISES

1. Two pyramids have congruent bases and congruent lateral faces arranged in the same order about the respective vertices. Show that the pyramids are congruent.

2. If the order of arrangement of the lateral faces of one pyramid in Ex. 1 is the reverse of the order in the other, show that the pyramids have equal volumes.

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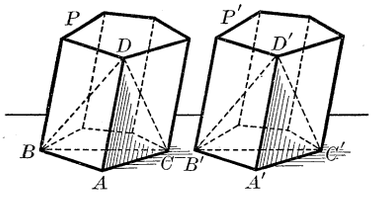
PROPOSITION II. THEOREM

870. *If two convex polyhedrons have their faces respectively congruent and arranged in the same order, they are congruent.*



Given the polyhedrons P and P', whose faces are respectively congruent and arranged in the same order.

To prove *that* $P \equiv P'$.

Proof. 1. Assume that A and A' are the vertices of homologous trihedral angles.*

Draw the homologous diagonals BC, CD, DB, and $B'C'$, $C'D'$, $D'B'$ in the faces adjoining these trihedral angles.

2. Then $\triangle ABC \equiv \triangle A'B'C'$,

$\triangle ACD \equiv \triangle A'C'D'$,

and $\triangle ABD \equiv \triangle A'B'D'$.

$\therefore$ tetrahedron $ABCD \equiv$ tetrahedron $A'B'C'D'$, § 868

and $\triangle BCD \equiv \triangle B'C'D'$.

3. If $ABCD$ and $A'B'C'D'$ are removed, the hypothesis holds for the remaining portions of P and P', and proceeding as before it can be shown finally that P and P' consist of tetrahedrons congruent in pairs and similarly placed.

4. $\therefore$ all the parts (edges, face angles, and dihedral angles) are respectively equal and arranged in the same order, and P and P' can be made to coincide.

That is, $P \equiv P'$.

*In case there is at any stage in the following proof no trihedral angle at any vertex, the proof is too long for the purposes of an elementary text.

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871. Corollary 1. *Two prisms are congruent if the three faces which include a trihedral angle of the one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed.*

Superpose the given congruent faces including the trihedral angles (which are now congruent by § 819), and then prove that all the faces are respectively congruent and similarly placed.

872. Corollary 2. *Two congruent convex polyhedrons can be separated into the same number of tetrahedrons, congruent each to each and similarly placed.*

873. Corollary 3. *Two convex polyhedrons symmetric with respect to a point or a plane are equal in volume.*

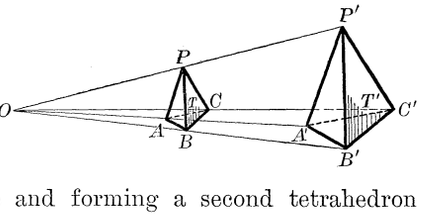
They can be broken up into the same number of symmetric tetrahedrons. But these tetrahedrons are respectively equal (§ 869).

874. From the above theorem (§ 870) it follows that

A convex polyhedron is determined when its faces are determined.

875. Similar Polyhedrons. It has already been shown (§ 660) that a section of a pyramid parallel to the base is a polygon similar to the base.

Given a tetrahedron $PABC$. Suppose that rays are drawn from any convenient common origin O through the vertices of the given tetrahedron, forming a polyhedral angle; and suppose that planes are passed parallel to the faces of $PABC$, cutting the faces of the polyhedral angle and forming a second tetrahedron $P'A'B'C'$ whose edges lie in the faces of the polyhedral angle. The new tetrahedron thus determined has faces which are respectively similar to the faces of the given tetrahedron and arranged in the same order. The corresponding dihedral angles of the tetrahedrons will be equal, since their faces are in



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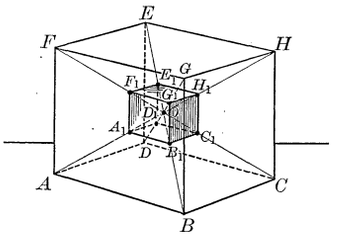
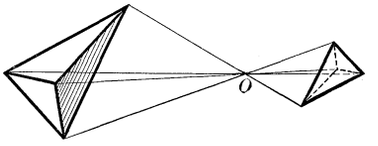
each case respectively parallel and lie on the same side of that face of the polyhedral angle which passes through the edges.

A corresponding construction can be used to derive from any given polyhedron a second polyhedron whose faces are respectively similar to those of the given polyhedron, and similarly placed.



Two such polyhedrons are called **similar polyhedrons.** And since a convex polyhedron is determined by its faces (§ 874), two similar convex polyhedrons may be defined as convex polyhedrons whose faces are similar each to each and arranged in the same order.

876. If the second polyhedron is determined by planes which are on the opposite side of O from the faces of the given polyhedron, the polyhedrons are called **inversely similar,** since their faces are similar each to each, but arranged in reverse order.



Preliminary Theorems

877. *A section of a pyramid parallel to the base cuts off a pyramid similar to the given pyramid.*

878. *Homologous edges of similar polyhedrons are proportional.*

879. *Two homologous faces of similar polyhedrons are proportional to the squares on any two homologous edges.* (§ 416.)

880. *The total surfaces of two similar polyhedrons are proportional to the squares on any two homologous edges.* (Apply § 361.)

881. The **ratio of similitude** of two similar polyhedrons is the ratio of two homologous edges. In the construction of § 875 the ratio of the distances of two corresponding vertices from the origin is equal to the ratio of similitude. (Why?)

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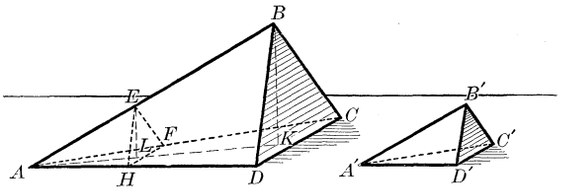
PROPOSITION III. THEOREM

882. *The volumes of two similar tetrahedrons are to each other as the cubes of any two homologous edges.*



Given the similar tetrahedrons *ABCD* and *A′B′C′D′* with volumes *v* and *v′*, and the homologous edges *AB* and *A′B′*, *AC* and *A′C′*, *AD* and *A′D′*.

To prove *that* $$\frac{v}{v'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. 1. On AB take $AE = A'B'$, on AC take $AF = A'C'$, and on AD take $AH = A'D'$. Draw EF, FH, and EH.

Then tetrahedron $AEHF \equiv$ tetrahedron $A'B'C'D'$. § 868

2. Let AK be the orthogonal projection of AB upon the plane ACD. Draw BK and $EL \perp$ to AK.

3. Then BK and EL are the altitudes of the tetrahedrons $ABCD$ and $AEFH$ respectively. § 569

$$\therefore \frac{v}{v'} = \frac{\triangle ADC \cdot BK}{\triangle AHF \cdot EL}. \qquad \text{§ 712}$$

4. But $$\frac{\triangle ADC}{\triangle AHF} = \frac{\overline{AC}^2}{\overline{AF}^2}. \qquad \text{§ 416}$$

5. And in the similar $\triangle ABK$ and AEL

$$\frac{BK}{EL} = \frac{AB}{AE}.$$

6. Also, in the similar $\triangle ABC$ and AEF

$$\frac{AC}{AF} = \frac{AB}{AE}.$$

$$\therefore \frac{v}{v'} = \frac{\overline{AB}^3}{\overline{AE}^3} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

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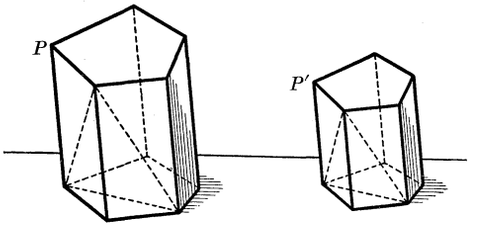
PROPOSITION IV. THEOREM

883. *Two similar convex polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.*



Given the similar convex polyhedrons P and P'.

To prove *that P and P' can be separated into the same number of tetrahedrons similar each to each and similarly placed.*

Suggestion. Choose a convenient origin O and construct as in § 875 a polyhedron Q similar to P' and with a ratio of similitude (§ 881) equal to that of P and P'. Then $Q \equiv P$ (§ 870). Divide Q and P into the same number of tetrahedrons congruent each to each and similarly placed (§ 872). The foregoing construction can now be applied to the tetrahedrons in Q to divide P' into tetrahedrons.

884. COROLLARY. *The volumes of two similar convex polyhedrons are to each other as the cubes of any two homologous edges.*

For the polyhedrons can be separated into the same number of tetrahedrons, similar each to each, whose volumes are respectively proportional to the cubes of their homologous edges, and therefore to the cubes of any two homologous edges of the given polyhedrons (AX. 1). Hence their sums have the same ratio (§ 361).

885. Similar cylinders of revolution are cylinders which are generated by revolving similar rectangles about homologous sides as axes.

886. Similar cones of revolution are cones which are generated by revolving similar right triangles about homologous legs as axes.

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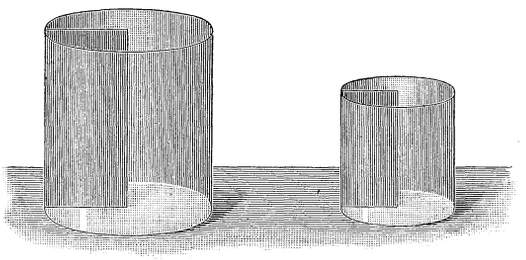
PROPOSITION V. THEOREM

887. *The lateral areas or the total areas of similar cylinders of revolution are to each other as the squares of their altitudes or as the squares of their radii, and their volumes are to each other as the cubes of their altitudes or as the cubes of their radii.*



Given two similar cylinders of revolution, with lateral areas l and l', total areas t and t', volumes v and v', altitudes h and h', and radii r and r'.

To prove *that* $l : l' = t : t' = h^2 : h'^2 = r^2 : r'^2$,

and that $v : v' = h^3 : h'^3 = r^3 : r'^3$.

Proof. 1. Since the generating rectangles are similar,

$$\frac{h}{h'} = \frac{r}{r'} = \frac{h + r}{h' + r'}.$$ Why?

2. Also $$\frac{l}{l'} = \frac{2\pi rh}{2\pi r'h'} = \frac{rh}{r'h'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

3. But $$t = 2\pi rh + 2\pi r^2 = 2\pi r(h + r).$$

$$\therefore \frac{t}{t'} = \frac{2\pi r(h + r)}{2\pi r'(h' + r')} = \frac{r(h + r)}{r'(h' + r')} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

4. Also $$v = \pi r^2 h.$$

$$\therefore \frac{v}{v'} = \frac{\pi r^2 h}{\pi r'^2 h'} = \frac{r^2 h}{r'^2 h'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}.$$

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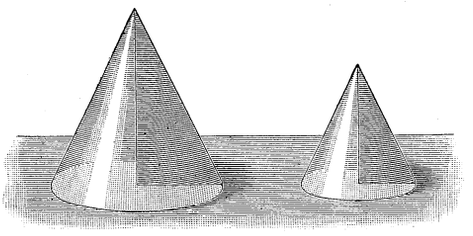
PROPOSITION VI. THEOREM

888. *The lateral areas or the total areas of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their radii, as the cubes of their altitudes, or as the cubes of their slant heights.*



Given two similar cones of revolution, with lateral areas l and l', total areas t and t', volumes v and v', altitudes h and h', radii r and r', and slant heights s and s' respectively.

To prove *that* $l : l' = t : t' = h^2 : h'^2 = r^2 : r'^2 = s^2 : s'^2$,

and that $v : v' = h^3 : h'^3 = r^3 : r'^3 = s^3 : s'^3$.

Proof. 1. Since the generating triangles are similar,

$$\frac{h}{h'} = \frac{r}{r'} = \frac{s}{s'} = \frac{s + r}{s' + r'}.$$ Why?

2. Also $\frac{l}{l'} = \frac{\pi rs}{\pi r's'} = \frac{rs}{r's'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{h^2}{h'^2}$.

3. Also $t = \pi rs + \pi r^2 = \pi r(s + r)$.

$$\therefore \frac{t}{t'} = \frac{\pi r(s + r)}{\pi r'(s' + r')} = \frac{r(s + r)}{r'(s' + r')} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{h^2}{h'^2}.$$

4. Also $v = \frac{1}{3}\pi r^2 h$.

$$\therefore \frac{v}{v'} = \frac{\pi r^2 h}{\pi r'^2 h'} = \frac{r^2 h}{r'^2 h'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3}.$$

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EXERCISES

1. Show that two unequal cubes are similar; that two unequal regular tetrahedrons are similar.

2. The dimensions of a box are 4, 5, and 6. What will be the dimensions of a similar box which holds 27 times as much? 5 times as much?

3. What is the ratio of the total areas of the boxes mentioned in each case in Ex. 2?

4. The bases of two similar pyramids are to each other as 4 : 25. What is the ratio of their altitudes? of their volumes?

5. The altitudes of two similar pyramids are to each other as 3 : 5. What is the ratio of their bases? of their volumes?

6. By what number must the radius and altitude of a cylinder of revolution each be multiplied in order to determine a similar cylinder whose volume is

(a) 8 times as great?
(b) 27 times as great?
(c) 5 times as great?
(d) $\frac{1}{8}$ as great?
(e) $\frac{1}{2}$ as great?
(f) $1/n$ as great?

7. By what number must the radius and altitude of a cylinder each be multiplied to determine a similar cylinder whose total area is

(a) 4 times as great?
(b) 6 times as great?
(c) n times as great?
(d) $\frac{1}{9}$ as great?
(e) $\frac{1}{2}$ as great?
(f) $1/n$ as great?

8. The mid-section of a pyramid has what fraction of the volume of the pyramid above it? what fraction of the lateral area?

9. At what distances from the vertex must a pyramid (cone) 6 ft. high be cut by planes parallel to the base in order to divide the pyramid (cone) into three equal parts? 4 equal parts? n equal parts?

10. At what distance from the vertex of a pyramid 10 ft. high must a plane be passed to divide the pyramid into two parts which shall be in the ratio 3 : 4? in the ratio $m : n$? into three parts which shall be in the ratio $a : b : c$?

11. At what distance from the vertex must a plane be passed parallel to the base of a pyramid to divide the lateral area in half? in the ratio 2 : 3? in the ratio $m : n$?

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Table of Squares, Cubes, Square Roots, and Cube Roots of Numbers

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

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Trigonometric Table

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1.072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.43
42°	.669	.743	.900	86°	.9976	.070	14.30
43°	.682	.731	.933	87°	.9986	.052	19.08
44°	.695	.719	.966	88°	.9994	.035	28.64
45°	.707	.707	1.000	89°	.9998	.017	57.29

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